MATHEMATICAL

QUESTIONS AND SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

PUBLISHED IN THE "EDUCATIONAL TIMES,"

AND

AN APPENDIX.

EDITED BY

W. J. C. MILLER, B.A.,

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Solved Questions.

APPENDIX.

ERRATUM.

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MATHEMATICS

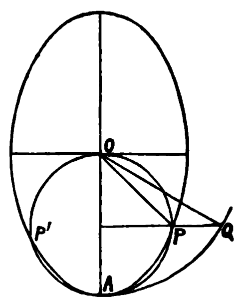
FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

5517. (H. Pollexfen, B.A.)—An ellipse is placed with its major axis vertical; find the semi-diameter of quickest descent, and the condition that it may be the major axis.

Solution by H. J. Woodall, A.R.C.S.



Draw the circle with the lower part (OA) of the semi-major axis of the ellipse as diameter. This circle will touch at A, and if it cuts the ellipse, will cut the ellipse at two points P, P′, equidistant from O. Join OP, then OP will be the semi-diameter of quickest descent. Let the point Q (on the auxiliary circle) correspond to P. Let

$$\angle \text{AOP} = \theta, \quad \text{AOQ} = \phi, \quad \text{OA} = a.$$

Since P is on the ellipse

$$\text{OP}^2 = a^2\cos^2\phi + b^2\sin^2\phi = a^2\cos^2\theta.$$

Now project OQ, OP on the vertical diameter;

therefore $\qquad a^2\cos^2\phi = a^2\cos^4\theta;$

therefore $\qquad \cos\phi = \cos^2\theta, \quad \sin^2\phi = 1-\cos^4\theta;$

$\therefore\; a^2\cos^4\theta + b^2(1-\cos^4\theta) = a^2\cos^2\theta;\;\; \therefore \cos^2\theta = b^2/(a^2-b^2) = (1-e^2)/e^2;$

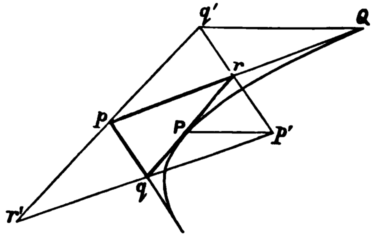
therefore $\qquad \cos\theta = (1-e^2)^{\frac{1}{2}}/e$

gives angle θ. When the vertical major axis coincides with the semi-diameter of quickest descent, we have $\cos\theta = 1$; therefore $1/e = \sqrt{2}$.

11918. (J. Griffiths, M.A.)—Through each angular point of any triangle circumscribing a parabola a line is drawn parallel to the opposite side; prove that the new triangle formed by these three lines is self-

conjugate with respect to the parabola. Hence show that the nine-point circle of any triangle self-conjugate with respect to a parabola passes through the focus, and that the centre of its circumscribing circle lies on the directrix.

Solution by Rev. J. J. MILNE, M.A.; H. W. CURJEL, B.A.; *and others.*



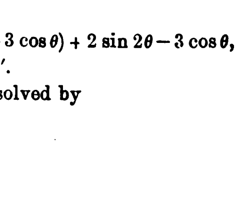
Let pqr be the tangent triangle, $p'q'r'$ the triangle formed by the parallels. Since $qprp'$, $qpq'r$ are parallelograms, r is the mid-point of $p'q'$, and so for p and q. Join Pp'. Then by *Parabola* (MILNE and DAVIS), Art. 33, Pp' is a diameter, and by Art. 36, $q'r'$ is the polar of p'. Similarly Qq' is a diameter, and $p'r'$ is the polar of q'. Therefore $p'q'r'$ is self-conjugate with respect to the parabola.

(2) Since p, q, r are the mid-points of the sides of $p'q'r'$, the nine-point circle of $p'q'r'$ is the circumcircle of pqr, and therefore, by Art. 41, passes through the focus.

(3) The circumcentre of $p'q'r'$ is obviously the orthocentre of pqr, and therefore, by Art. 73, lies on the directrix.

10446. (Professor NRITYA GOPAL SARKAR.)—Two florins with smooth rims are placed in the corner between two smooth vertical planks. A sixpenny-piece can be pressed between these coins without causing them to separate. Find the angle between the vertical planks.

Solution by H. J. WOODALL, A.R.C.S.



Let the figure represent a horizontal section through point of contact of florins (F, F′). Let P, P′ be the "planks," S the sixpence.

Let the angle be 2θ, $2a$ = thickness of sixpence, breadth $= b$, c = thickness of florin;

then $AC = a\cot\theta + b + a\tan\theta = c\operatorname{cosec}\theta$;

$$\therefore\ a + b\sin\theta\cos\theta = c\cos\theta.$$

We have $2a = \frac{1}{24}$ inch, $b = \frac{109}{144}$ inch, $c = \frac{13}{192}$; the equation becomes

$0 = 12 + 218\sin 2\theta - 39\cos\theta \equiv 12(1 + 18\sin 2\theta - 3\cos\theta) + 2\sin 2\theta - 3\cos\theta$,

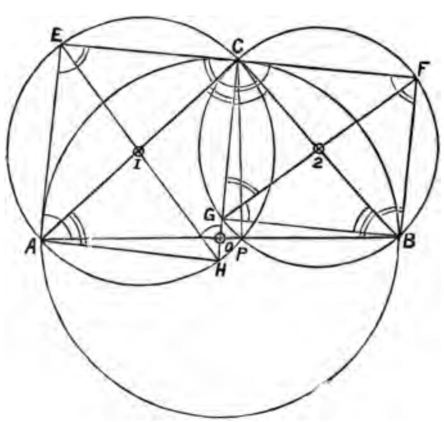
which will be satisfied if we put $\theta = 3°32'50{\cdot}5''$.

The simpler form $0 = 1 + 18\sin 2\theta - 3\cos\theta$ is solved by

$$\theta = 3°\,10'\,56{\cdot}2''.$$

4058. (A. RENSHAW.)—Let ACB be a right-angled triangle, and CP the perpendicular on AB. Then, if circles be drawn on the three sides, and a tangent to that on AB be drawn through C, cutting the other two circles in E and F, prove (1) that CF shall be equal to CE; and (2) that the line FCE shall divide the exterior semicircles on AC, CB into four segments, the sum of the alternate pairs of which shall be respectively equal to the segments of the semicircle on AB made by AC, CB; also (3) deduce other properties of the figure.

Solution by H. J. WOODALL, A.R.C.S.



Let O, O_1, O_2 be mid-points of AB, AC, BC respectively.

(1) Because E is on circle drawn on AC as diameter. Therefore AEC = right angle = CFB = ECO (because EC tangential to circle AB at C). But AO = OB, and therefore EC = CF.

(2) Segment on AE = multiple of $AO_1^2 \times \angle AO_1E = K_1 \times AO_1^2 \times \angle AO_1E$. So segments an EC, CF, FB are

$K_2 \times AO_1^2 \times \angle EO_1C$,

$K_1 \times CO_2^2 \times \angle CO_2F$,

and $K_2 \times CO_2^2 \times \angle FO_2B$.

Also $AO_1E = CO_2F = AOC$, and $EO_1C = FO_2B = COB$. Therefore segments on AE and CF $= K_1 \times \angle AO_1E \times (AO_1^2 + CO_2^2) = K_1 \times \angle AOC \times AO^2$; and so segments on EC, FB $= K_2 \times \angle COB \times AO^2$.

(3) Circles on AC, CB (as described) pass through P. If CO cuts these circles in G and H, then OG . OC = OP . OB. Hence GP is anti-parallel to CB in $\angle$ BOC. The angles in the figure are marked with one or two lines, according as to which of two values they have.

Again, OO_2, CP, and GB are concurrent at L (because OC, OB, LC, LB, O_2C, O_2B are equal in pairs). So also AH and CP meet on O_1O.

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650. (J. WILLIAMS, M.A.)—In the base AB of a triangle find geometrically a point P, such that by joining it with two given points Q, R by right lines intersecting the other two sides AC, CB, at *m* and *n*, the figure A*mn*B shall be a minimum.

Solution by D. BIDDLE.

The above question can best be answered by considering first the position of P when AmnB is a maximum. This requires that the triangle Cmn should be a minimum, which occurs when Cm . Cn is least; and this, when

$$Cm \,.\, Cn = (Cm + dx)\,(Cn - k \,.\, dx),$$

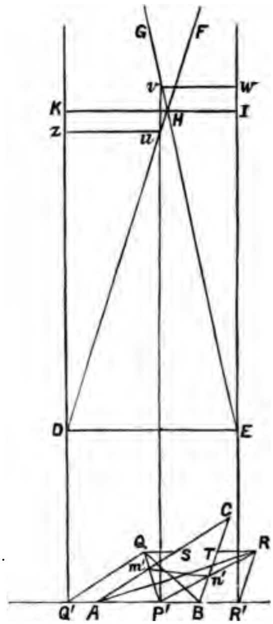
where $\quad k = Cn/Cm$;

that is to say, when the infinitesimal increment and decrement are as the respective distances from C. Draw QS, RT parallel to AB; also QQ′, RR′ parallel to AC, CB, meeting AB produced in Q′, R′. If we now regard P when in the required position (for the reverse of the question), and also when moved along AB through an infinitesimal distance, we obtain the following equation:—

$$Qm/QP \,.\, Am/AP : Rn/RP \,.\, Bn/BP = Cm : Cn \;\ldots\ldots\ldots\; (\alpha);$$

the terms on the left being proportionate to the infinitesimal movements of m and n in AC and CB respectively, for all positions of P in AB. But

$$Qm'/QP' = QS/Q'P';$$
$$Am'/AP' = AS/Q'P';$$
$$Rn'/RP' = RT/R'P';$$
$$Bn'/BP' = BT/R'P'.$$



Consequently the proportion between the infinitesimal movements of m and n (dx and dy) may be represented thus:—

$$dx : dy = QS \,.\, AS/(Q'P')^2 : RT \,.\, BT/(R'P')^2 \;\ldots\ldots\ldots\ldots\; (\beta).$$

Moreover, $Cm' = CS + QS \,.\, AS/Q'P'$; and $Cn' = CT + RT \,.\, BT/R'P'$. Wherefore, by reduction of (α), we obtain

$$\frac{CS}{QS \,.\, AS}(Q'P')^2 + Q'P' = \frac{CT}{RT \,.\, BT}(R'P')^2 + R'P' \;\ldots\ldots\ldots\; (\gamma).$$

Let $Q'R' =$ unity, $CS/(QS \,.\, AS) = \mu$, $CT/(RT \,.\, BT) = \nu$. Then from (γ) we obtain $1 + \mu Q'P' : 1 + \nu(1 - Q'P') = 1 - Q'P' : Q'P'$. On Q′R′ erect a square, produce its vertical sides, and with its upper side form the angles EDF, DEG, having μ, ν for their respective tangents. It is then easy to draw a straight line perpendicular to Q′R′, cutting DF, EG in u, v, such that $P'u : P'v = vw : uz$; for

$$Q'P' = \left\{(1 + \mu + \nu + \mu\nu)^{\frac{1}{2}} - (1 + \nu)\right\}/(\mu - \nu),$$

and this can be rendered geometrically by the method now become familiar to readers of our pages.

On turning to the question proper, it is clear that P must be on one or other side of P′; it must also be within AB, to make AmnB fitly designate the figure formed; and $(\mathrm{C}m+x)(\mathrm{C}n-y)$ must be a maximum. Now, as P moves along AB from P′, the elements dx, dy, which make up x, y respectively, bear to each other, according to (β), the ratio, varying at every stage, $\mathrm{QS}\,.\,\mathrm{AS}/(\mathrm{Q'P})^2 : \mathrm{RT}\,.\,\mathrm{BT}/(\mathrm{R'P})^2$. As P recedes from P′ towards A, the succeeding elements of x are larger, those of y are smaller; as it recedes towards B, the reverse is the case. Considering the former more particularly, let dx, dy be the initial elements, where $dy = k\,.\,dx = \mathrm{C}n/\mathrm{C}m\,.\,dx$; and let p_1dx, q_1dy, p_2dx, q_2dy, &c., be the succeeding ones, where $1 < p_1 < p_2 <$ &c., and $1 > q_1 > q_2 >$ &c. Then we have

$$\{\mathrm{C}m + (1+p_1)dx\}\,.\,\{\mathrm{C}n-(1+q_1)dy\} > (\mathrm{C}m+dx)(\mathrm{C}n-dy), \text{ by } \mathrm{C}n(p_1-q_1)dx.$$

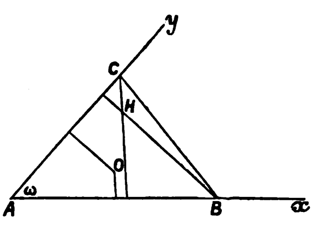
Similarly, it can be shown that

$$\{\mathrm{C}m+(1+p_1+p_2)\,dx\}\,.\,\{\mathrm{C}n-(1+q_1+q_2)\,dy\}$$
$$> \{\mathrm{C}m+(1+p_1)\,dx\}\,.\,\{\mathrm{C}n-(1+q_1)\,dy\}, \text{ by } \mathrm{C}n\,(p_2-q_2)\,dx,$$

and so on. Consequently the triangle Cmn increases with acceleration, until P reaches A (or B). To determine which has the advantage, we must observe whether AR or BQ has its intersection (at n or m) nearest AB. In the adjoining figure, A has the advantage. Therefore A and P and also m coincide in that instance.

11628. (Professor Desprez.)—Les côtés de l'angle A d'un triangle ABC sont fixes; le côté BC roule sur une courbe donnée Δ. Démontrer que l'orthocentre H du triangle ABC, et le centre O du cercle circonscrit, décrivent deux figures symétriquement semblables.

Solution by H. W. Curjel, B.A.; Professor Droz Farny; *and others.*



Let x_1, y_1, x_2, y_2 be the coordinates of H and O referred to AB, AC as axes of x and y. Let $\angle\,\mathrm{BAC} = \omega$. Let $\mathrm{AC} = b$, $\mathrm{AB} = c$; then

$$x_1 = \frac{\cos\omega}{1-\cos^2\omega}\,(b-c\cos\omega),$$

$$y_1 = \frac{\cos\omega}{1-\cos^2\omega}\,(c-b\cos\omega),$$

$$x_2 = \tfrac{1}{2}y_1\sec\omega, \quad y_2 = \tfrac{1}{2}x_1\sec\omega.$$

Now, if BC envelopes a given curve, we have an equation $f(b, c) = 0$ connecting b and c; hence H and O will describe two curves, symmetrically similar, since

$$x_2 = \tfrac{1}{2}y_1\sec\omega, \quad y_2 = \tfrac{1}{2}x_1\sec\omega.$$

11874. (S. Tebay.)—If three of the numbers 1, 2, 3, ... n be taken at random, find the probability that every two of them shall be greater than the third: and show that, if n be indefinitely large, the chance ultimately approaches the limit $\frac{3}{8}$.

Solution by the Proposer.

Let x, y, z be three of the numbers taken in order. It is only necessary to consider the cases in which $x+y>z$. As a limit, take $x+y-1=z$. We thus see that between y and z there are generally $x-1$ numbers, each of which furnishes $x-1$ cases. The least of these numbers is $n-x+1$; and between x and $n-x+1$ there are $n-2x+1$ numbers, each furnishing $x-1$ favourable cases, or $(x-1)(n-2x+1)$ in all.

Let $n=2m$; then x can have all values from 2 to m; and it is found on summation that there are $\frac{1}{6}m(m-1)(2m-1)$ favourable cases. There are still m numbers remaining which do not follow this law, but it is plain that every three of them answer the conditions of the problem. This number is $\frac{1}{6}m(m-1)(m-2)$.

Hence the number of favourable cases is

$$\tfrac{1}{6}m(m-1)(2m-1)+\tfrac{1}{6}m(m-1)(m-2)=\tfrac{1}{2}m(m-1)^2.$$

And, since the whole number of cases is

$$\tfrac{1}{6}2m(2m-1)(2m-2)=\tfrac{2}{3}m(m-1)(2m-1),$$

the probability required, when n is even, is

$$\frac{\frac{1}{2}m(m-1)^2}{\frac{2}{3}m(m-1)(2m-1)}=\frac{3}{4}\cdot\frac{m-1}{2m-1}=\frac{3}{8}\cdot\frac{n-2}{n-1}.$$

Let $n=2m+1$. We have already considered the case in which $n=2m$; the increment on the introduction of the number $2m+1$ will consist of terms in which $2m+1$ is united with every two of the $2m$ numbers whose sum is greater than $2m+1$. Let x be any number not greater than m. Then $2m+2-x$ is the least number which, added to x, exceeds $2m+1$: there are, therefore, $2m+1-(2m+2-x)=x-1$ available numbers, which produce $\frac{1}{2}x(x-1)$ cases, or $\frac{1}{2}m(m-1)$ in all. There are still $m-1$ terms remaining, the sum of every two of which is greater than $2m+1$. This number is $\frac{1}{2}m(m-1)$. Therefore the increment is $m(m-1)$; and $\frac{1}{2}m(m-1)^2+m(m-1)=\frac{1}{2}m(m^2-1)$. Hence the chance in this case is

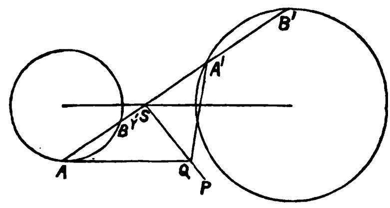
$$\frac{\frac{1}{2}m(m^2-1)}{\frac{1}{6}(2m+1)(2m)(2m-1)}=\frac{3}{2}\cdot\frac{m^2-1}{4m^2-1}=\frac{3}{8}\cdot\frac{(n+1)(n-3)}{n(n-2)}.$$

When n is indefinitely large, the limit of these results is $\frac{3}{8}$.

11467. (Madame F. Prime.)—Une droite Δ, passant par le centre de similitude interne S de deux circonférences C et C′, est coupée respectivement par ces deux circonférences aux quatre points A, B, A′, B′, que l'on prend comme centres de quatre circonférences ayant S pour point commun. La tangente commune extérieure DD′ aux deux circonférences A, A′ ren-

contre la tangente commune extérieure EE′ aux deux circonférences BB′, en un point M, dont on demande le lieu lorsque la sécante Δ varie.

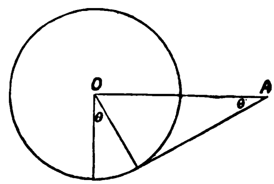
Solution by J. H. Grace.



Through S draw a line SP perpendicular to Δ. Then, if the common tangent to A, A′ meet SP in Q, it is evident that AQA′ is a right angle; therefore $SQ^2 = AS.SA'$, and, since $AS.SA' = BS.SB'$, the external common tangent to B, B′ (which is on the same side of Δ) must pass through Q, or Q is the point of which the locus is sought. Now $SQ^2 = AS.SA'$, which is constant for all positions of Δ; therefore SQ is constant, and the locus of Q is a circle having S for centre.

11853. (Professor Clifford, F.R.S.)—Three elastic strings without weight, whose natural lengths are OA, OB, OC, are joined together at O, the centre of the circumscribing circle of the horizontal triangle ABC; and a smooth sphere of given radius and weight is placed with its centre vertically above O, and allowed to descend until the centre rests at O. Find the moduli of elasticity in the three strings.

Solution by H. W. Curjel, B.A.; W. J. Dobbs, B.A.; *and others*.



Let T_a, T_b, T_c be the tensions of the strings, W = the weight of the sphere, r its radius, R the radius of circle ABC.

Let $\theta = \sin^{-1} r/R$; then

$$\frac{T_a}{\sin 2A} = \frac{T_b}{\sin 2B} = \frac{T_c}{\sin 2C} = \frac{(T_a + T_b + T_c)\sin\theta}{\sin\theta\Sigma\sin 2A} = \frac{W}{\sin\theta\Sigma\sin 2A} = \frac{WR}{r\Sigma\sin 2A}.$$

Extension of each of the strings $= (R^2 - r^2)^{\frac{1}{2}} + r\theta - R$, therefore modulus of elasticity of OA $= E_a = \dfrac{RT_a}{(R^2 - r^2)^{\frac{1}{2}} + r\theta - R}$

$$= \frac{WR^2 \sin 2A}{\left\{r\sin^{-1} r/R + (R^2 - r^2)^{\frac{1}{2}} - R\right\} r\Sigma\sin 2A},$$

and we have similar expressions for E_b, E_c.

6379. (Professor TANNER, M.A.)—If $S_n = \frac{1}{m} + \frac{1}{m-1} + \dots + \frac{1}{m-n+1}$, find the value of n such that (m being not necessarily integral)

$$S_n < 1 < S_{n+1} \text{ or } S_{n+1} = 1.$$

Solution by H. J. WOODALL, A.R.C.S.

If we put $u_x = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + 1/x$,

then $S_n = u_m - u_{m-n} = C + \log m + \frac{1}{2m} - \frac{1}{12m^2} + \frac{1}{120m^4} - \&c.$

$$- \left\{ C + \log(m-n) + \frac{1}{2(m-n)} - \frac{1}{12(m-n)^2} + \frac{1}{120(m-n)^4} - \&c. \right\}$$

$$= \log \frac{m}{m-n} + \frac{1}{2}\left(\frac{1}{m} - \frac{1}{m-n}\right) + \frac{1}{12}\left(\frac{1}{(m-n)^2} - \frac{1}{m^2}\right) - \frac{1}{120}\left(\frac{1}{(m-n)^4} - \frac{1}{m^4}\right)$$

$= y$, where y is equal to or less than 1.

This is not difficult of solution, since n must be an integer.

The converse problem, to find the least values of m corresponding to successive integral values of n, includes the former.

$n=1$	2	3	4	5	6	7	8	9	10	11	12
$m=1$	2·618034	4·22	5·80	7·39	8·97	10·56	12·14	13·73	15·31	16·89	18·47

[The question arises, whether the differences of m (for unit differences of n) are asymptotic to a number which may be put provisionally at 1·57; and, if not, why. The asymptotic value is $e/(e-1)$, $= 1·582$.]

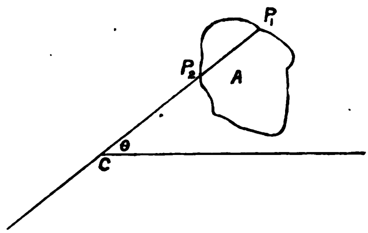
11867. (Professor CESARO.)—Les parallèles menées par un point de l'ellipse de Steiner aux médianes du triangle rencontrent les côtés opposés sur une droite. Etudier l'enveloppe de cette droite, lorsque le point se déplace sur l'ellipse.

Solution by Professors DROZ FARNY, BHATTACHARYA, *and others.*

Considérons un triangle équilatéral inscrit dans une circonférence. Les parallèles aux médianes de ce triangle par un point quelconque P de la circonférence coupent les côtés opposés en trois points situés sur la ligne de Simson du point P. L'enveloppe de cette droite est l'hypocycloide de Steiner. Projetons la figure orthogonalement sur un plan; le triangle devient quelconque, la circonférence devient l'ellipse de Steiner, et la courbe demandée est une quartique, projection orthogonale d'une hypocycloide de Steiner.

7937. (By Hugh McColl, B.A.)—Several areas are scattered anyhow on a plane. From any point C in this plane a straight line is drawn at random, meeting the boundaries of one or more of these areas, each at the point P. Show that the average area of the circle whose radius is CP is equal to the sum of the several areas, the circle to be taken positively when P is a point of exit, negatively when P is an entrance point, and zero when there is no P, in other words, when the random line meets no boundary.

Solution by H. W. Curjel, B.A.



Let one of the straight lines CP meet one of the areas A in P_1P_2, P_2 being an entrance point, and P_1 an exit point. Let $CP_1 = r_1$, $CP_2 = r_2$.

Then mean value

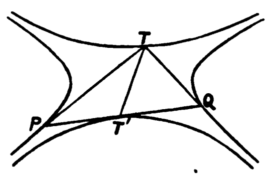
$$= \Sigma \int \pi \left(\frac{r_1^2 - r_2^2}{2}\right) d\theta \Big/ \int_0^\pi d\theta,$$

the integral being taken over all the area A.

$$= \pi \Sigma A / \pi = \Sigma A = \text{sum of the several areas.}$$

11877. (Rev. T. Roach, M.A.) — Tangents are drawn to a hyperbola from any point T on the conjugate, and a chord is drawn through the centre parallel to the tangent at T; prove that the chord bisects the tangents.

Solution by W. J. Dobbs, M.A.; Professor Zerr; *and others.*



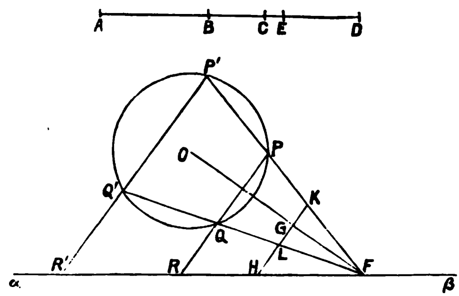
Let TP, TQ be the tangents, T′ the other extremity of the diameter through T.

By a well-known theorem, the polar of T with respect to the original hyperbola is the tangent at T′; *i.e.*, PQ is the tangent at T′.

Therefore, &c.

11841. (Morgan Brierley.)—A circle is given in magnitude and position; also a straight line is given in position; it is required to draw a chord in the circle, which, produced, shall meet the straight line at a given angle, at a point to be determined, so that the part of the chord line produced shall have a given ratio to a tangent drawn to the circle from the same point.

Solution by W. J. DOBBS, B.A.; Professor ZERR, M.A.; *and others.*

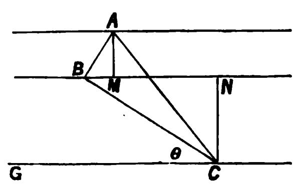


Let O be the centre of the given circle, $\alpha\beta$ the given straight line, and AB : AC the given ratio. Take AD, third proportional to AB, AC, and bisect BD in E. From O draw OF perpendicular to given direction to meet $\alpha\beta$ in F, and from any point G in OF draw GH parallel to given direction to meet $\alpha\beta$ in H. Divide HG in K, L, externally and internally, in same ratio that AE is divided in D, B. Produce FK, FL to meet circle in PP′, QQ′.

Then either of chords PQR, P′Q′R′ is the chord required. For RQ : RP = AB : AD; hence RQ : tangent from R = AB : AC.

11668. (W. J. GREENSTREET, M.A.)—Find the diameter of the circumcircle of a triangle ABC, whose vertices lie on three parallel lines, distance m and n apart.

Solution by Madame THÉRÈSE; Professor ZERR; *and others.*



Let $\quad AM = m, \quad CN = n,$

$\angle BCG = \theta;$

then $\quad AB = m \operatorname{cosec}(B - \theta),$

$BC = n \operatorname{cosec}\theta;$

$$\frac{AB}{BC} = \frac{\sin C}{\sin A} = \frac{m \operatorname{cosec}(B-\theta)}{n \operatorname{cosec}\theta};$$

$$\therefore \quad \tan\theta = \frac{n \sin C}{m \sin A + n \sin C \cos B};$$

$$\text{diameter} = \frac{n \operatorname{cosec}\theta}{\sin A} = \frac{(m^2 \sin^2 A + n^2 \sin^2 C + 2mn \sin A \sin C \cos B)^{\frac{1}{2}}}{\sin A \sin B \sin C}.$$

4124. (Rev. A. F. TORRY, M.A.)—A certain transparent substance transmits only three of the colours of the spectrum, and these it absorbs partially and unequally. A ray of light traversing in succession a number

of equal plates of this substance, it is found that the ray emerging from the first plate consists half of the first colour, and of equal parts of the two others, whilst that emerging from the second plate consists half of the second colour and of equal parts of the two others. In what proportions will the colours be present in the ray which emerges from the third plate?

Solution by H. J. WOODALL, A.R.C.S.; Prof. ZERR; *and others.*

Call the colours x, y, z. Then there will be emerging from the first, second, and third plates, 2 of x, 1 of y, 1 of z; $2p$ of x, q of y, r of z: $2p^2$ of x, q^2 of y, r^2 of z. But $2p = \frac{1}{2}q = r$; therefore $q = 4p$, and the proportions of the colours emerging from the third plate are

$$2p^2 : 16p^2 : 4p^2 = 1 : 8 : 2.$$

11053. (Professor MUKHOPADHYAY, M.A.)—Describe a plane curve passing through given points in order, and enclosing the maximum area with a given perimeter.

Solution by H. W. CURJEL, B.A.

Join the given points A, B, C, &c., in order so as to form a closed polygon. The arc AaB of the curve, cut off by AB, must be the arc of a circle for the area to be a maximum; for, if it were not, we could draw a circular arc of the same length enclosing with the chord AB a greater area. Similarly, the arcs subtended by the other sides of the polygon are circular. Also, if the enclosed area is a maximum, it must be impossible to draw other arcs than AaB, BbC on AB, BC such that their sum = the sum of arcs AaB, BbC, and the sum of the areas contained by them and the chords AB, BC is greater than the sum of the areas contained by AaB and AB, and BbC and BC. Now, by turning BC round B till the tangents at B to the arc coincide, we see that for a maximum area AaBbC the arcs must form one continuous circular arc; for otherwise we could draw a circular arc = AaB + BbC enclosing with AC a greater area than AaBbC. Therefore the circular arcs AB, BC, CD, &c., must have the same curvature. This condition determines the curve completely, which is therefore composed of circular arc AB, BC, &c., all having the same radius.

8960. (W. J. C. SHARP, M.A.)—The most general expression for a spherical harmonic of order n (an integer) is $\Sigma \frac{n!}{k!\,l!\,m!} a_{k,l,m}\, x^k y^l z^m$, where $k+l+m = n$ and the coefficients satisfy all the equations of the form

$$a_{k+2,l,m} + a_{k,l+2,m} + a_{k,l,m+2} = 0$$

(where $k+l+m = n-2$) which can be formed for different integral values

of k, l, and m. Hence, if $(a, b, c, f, g, h \between x, y, c)^2$ be a spherical harmonic, $a+b+c=0$.

Solution by H. J. Woodall, A.R.C.S.

Taking the given expression as the representative term of the series, and operating on it by Laplace's operator $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$, we get

$$\Sigma \frac{n!}{k!\,l!\,m!} a_{k,l,m} \left\{k(k-1)x^{k-2}y^l z^m + l(l-1)x^k y^{l-2} z^m + m(m-1) x^k y^l z^{m-2}\right\}.$$

Now, select from the series the coefficient of $x^k y^l z^m$, and we get

$$(n+2)!\left\{a_{k+2,l,m}+a_{k,l+2,m}+a_{k,l,m+2}\right\}/(k!\,l!\,m!);$$

but, since the function (series) satisfies the equation $\nabla^2 u = 0$, we must have this coefficient of $x^k y^l z^m$ vanish.

Whence $$a_{k+2,l,m}+a_{k,l+2,m}+a_{k,l,m+2}=0,$$

where $k+l+m+2=n$; therefore $k+l+m=n-2$ (because the function is of the n^{th} order).

Applying this result to the general equation of the second order we find $a+b+c=0$.

11564. (Editor.)—Find the number of points of contact of two balls in (1) a triangular, (2) a rectangular pile; prove (3) that the first is never a square number; and find (4) when the second is a square.

Solution by H. J. Woodall, A.R.C.S.

(1) Number of contacts in base row $= n-1$; number of contacts between last two rows $= 2(n-1)$; therefore total number of contacts in base flat $= \Sigma 3(n-1) = \frac{3}{2}n(n-1)$, number of contacts between last two flats $= \frac{3}{2}n(n-1)$ [$=$ those in base flat]; therefore total number of contacts in pile $= \Sigma 3n(n-1) = (n-1)n(n+1) \neq$ square number .... (3).

(2) Rectangular pile $m \times n \times p$. Number of contacts in first row $= m-1$, between two rows $= m$; therefore total number of contacts in base is $(m-1)n + m(n-1)$, number of contacts between two flats $= mn$, and there are $(p-1)$ sets of these;

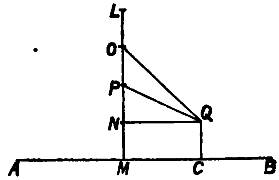
$\therefore$ total $= p\left\{(m-1)n+m(n-1)\right\} + mn(p-1) = 3pmn - (pm+mn+np)$.

(4) If $p=m=n$, this is $= p^2(3m-3)$; therefore $m-1$ must be equal to three times a square number.

11808. (I. Arnold.)—The point P and a straight line AB being given in position, find the locus of a point Q, so that the rectangle under QC (perpendicular on AB) and a given straight line m, shall be equal to the square on PQ.

Solution by PROPOSER; R. KNOWLES, B.A.; *and others.*

Let PM be the perpendicular on AB, and along MP produced take PL = m; the locus of Q is a circle whose centre is at O, the mid-point of PL.



$$\begin{aligned}OQ^2 &= OP^2+PQ^2+2\,OP\,.\,PN\\ &= OP^2+m\,.\,QC+2\,OP\,.\,PN\\ &= OP^2+2\,OP\,.\,NM+2\,OP\,.\,PN\\ &= OP^2+2\,OP\,.\,PM,\end{aligned}$$

which is fixed: *ergo.*

11869. (F. G. TAYLOR, M.A., B.Sc.)—If l, m, n be the distances of a point O from the vertices of a triangle ABC, prove that

$$a^2l^4+b^2m^4+c^2n^4-2bc\cos A m^2n^2-2\,ca\cos B\,.\,n^2l^2-2ab\cos C\,.\,l^2m^2$$
$$-2abc\,(a\cos A\,.\,l^2+b\cos B\,.\,m^2+c\cos C\,.\,n^2)+a^2b^2c^2=0.$$

Solution by Professor A. DROZ FARNY.

On a les formules connues

$$\cos\frac{BOC}{2}=\frac{1}{2}\Big/\frac{[(m+n+a)(m+n-a)]^{\frac{1}{2}}}{mn^{\frac{1}{2}}};$$

$$\cos\frac{AOB}{2}=\frac{1}{2}\Big/\frac{[(l+m+c)(l+m-c)]^{\frac{1}{2}}}{lm^{\frac{1}{2}}};$$

$$\cos\frac{AOC}{2}=\frac{1}{2}\Big/\frac{[(l+n+b)(l+n-b)]^{\frac{1}{2}}}{ln^{\frac{1}{2}}}.$$

Mais si 3 angles α, β, γ valent ensemble 180°, on a la relation

$$1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma=2\cos\alpha\cos\beta\cos\gamma;$$

d'où, en remplaçant les cosines par leurs valeurs respectives, cos α par cos $\frac{1}{2}$ (BOC) ... &c., on a, après avoir élevé au carré et développé :—

$$a^2l^4+b^2m^4+c^2n^4-(b^2+c^2-a^2)\,m^2n^2-(a^2+c^2-b^2)\,n^2l^2-(a^2+b^2-c^2)\,l^2m^2$$
$$-(b^2+c^2-a^2)\,a^2l^2-(a^2+c^2-b^2)\,b^2m^2-(a^2+b^2-c^2)\,c^2n^2+a^2b^2c^2.$$

Mais, d'après la relation $a^2=b^2+c^2-2bc\cos A$, on a

$$b^2+c^2-a^2=2bc\cos A.$$

Portant ces valeurs dans la relation précédente, on trouve immédiatement la relation de M. TAYLOR.

5324. (Dr. HART.)—Find two integral numbers, whose sum, difference, and difference of their squares shall each be a square, cube, and fourth power; and also the product of the nine roots of these powers shall be a square, cube, and fourth power.

Solution by Profs. ZERR, SARKAR, *and others.*

Let the numbers be

$$x^m + \frac{m(m-1)}{2} x^{m-2}y^2 + \ldots + \frac{m(m-1)}{2} x^2y^{m-2} + y^m,$$

$$mx^{m-1}y + \frac{m(m-1)(m-2)}{2\,.\,3} x^{m-3}y^3 + \ldots + \frac{m(m-1)(m-2)}{2\,.\,3} x^3y^{m-3} + mxy^{m-1}.$$

Then their sum is $(x+y)^m$, their difference $(x-y)^m$. The difference of their squares $(x^2-y^2)^m$, and the product of the nine roots $(x^2-y^2)^{\frac{1}{6}(13m)}$.

Now m must have such a value that m and $\frac{1}{6}(13m)$ are both divisible by 2, 3, and 4; this is the case when $m = 72$. Then the numbers are

$$x^{72} + 2556x^{70}y^2 + \ldots + 2556x^2y^{70} + y^{72},$$

$$72x^{71}y + 60480x^{69}y^3 + \ldots + 60480x^3y^{69} + 72xy^{71},$$

their sum is $(x+y)^{72}$, their difference $(x-y)^{72}$, the difference of their squares $(x^2-y^2)^{72}$, the product of the nine roots $(x^2-y^2)^{156}$, in which x, y have any value. Let $x = 2$, $y = 1$; then the numbers are

11264199772469587205920073937341321,

11264199772469587205920073937341320.

Their sum $= (3)^{72} = \{(3)^{36}\}^2 = (3^{24})^3 = (3^{18})^4$.

Their difference $= (1)^{72} = (1^{36})^2 = (1^{24})^3 = (1^{18})^4$.

Difference of squares $= (3)^{72} = (3^{36})^2 = (3^{24})^3 = (3^{18})^4$.

Product of nine roots $= (3)^{156} = (3^{78})^2 = (3^{52})^3 = (3^{39})^4$.

Any series of numbers can be found satisfying the conditions.

11745. (W. J. GREENSTREET, M.A.)—Find the locus of the centres of circles touching a conic so that the common tangents are always parallel.

Solution by PROPOSER, Professor SARKAR, *and others.*

The tangents from (α, β) to the ellipse $x^2/a^2 + y^2/b^2 = 1$ are

$$(x^2/a^2 + y^2/b^2 - 1)(\alpha^2/a^2 + \beta^2/b^2) = (x\alpha/a^2 + y\beta/b^2 - 1)^2.$$

If these are parallel,

$$(x^2/a^2 + y^2/b^2 - 1)(1/a^2 + m^2/b^2) = x/a^2 + my/b^2.$$

If the diameter of contact of one of the circles be $my + x + mn = 0$, and $\lambda = 1/(a^2b^2)$, the circle will be

$$(x^2/a^2 + y^2/b^2 - 1)(1/a^2 + m^2/b^2) - (x^2/a^2 + my/b^2)^2 + \lambda(my + x + mn)^2 = 0.$$

The lines of intersection of the ellipse and the circle are seen to be

$$(x/a^2 + my/b^2) = (my + x + mn)^2/(a^2b^2).$$

If $x/(am) + y/b + n/(a+b) = 0$ be a tangent at x_1, y_1, then

$$x_1 = a/(1+m^2)^{\frac{1}{2}}; \quad y_1 = bm/(1+x^2)^{\frac{1}{2}}.$$

The centre of the circle lies on the line $y = mx$ and on the normal to the ellipse at x_1, y_1; *i.e.*, upon $a\left[x(1+m^2)^{\frac{1}{2}}-a\right] = b\left[y(1+m^2)^{\frac{1}{2}}-bm\right]/m$.
Hence we find the centre locus to be, $x^2+y^2 = (a+b)^2$.

11622. (Professor Beyens.)—A point is taken at random on a given finite straight line of length a: prove that (1) the mean value of the sum of the squares on the two parts of the line is $\frac{2}{3}a^2$; and (2) the chance of the sum being less than this mean value is $\frac{1}{3}\sqrt{3}$.

Solution by H. W. Curjel, B.A.; Professor Zerr; *and others.*

(1) Mean value $= \frac{1}{a}\int_0^a \left\{x^2+(a-x)^2\right\}dx = \frac{2}{3}a^2$.

(2) Let the two numbers be expressed as $(b-x)$, $(b+x)$, where $2b = a$; then the sum of the squares $= 2(b^2+x^2)$.

If this is less than $\frac{2}{3}a^2$, $x^2 < \frac{1}{3}b^2$; but all values of x from 0 to b are equally likely. Hence the chance that the sum of the squares is less than the mean value is $\frac{1}{3}\sqrt{3}$.

11722. (Professor Wolstenholme, Sc.D.)—Prove that the mean value of $(x_1x_2x_3 \dots x_n)^{\frac{1}{2}}$, for all positive values of $x_1, x_2, \dots$ such that $x_1+x_2+\dots+x_n = 1$ is $\Gamma(r)\left\{\Gamma(\frac{3}{2})\right\}^n \div \Gamma(\frac{3}{2}n)$; and, more generally, that of $(x_1x_2\dots x_n)^{r-1}$, r being positive, is $\Gamma(n)\left\{\Gamma(r)\right\}^n \div \Gamma(nr)$.

Solution by H. W. Curjel, B.A.; Professor Zerr; *and others.*

The mean value of $(x_1x_2\dots x_n)^{r-1}$, where $x_1+x_2+x_3+\dots+x_n = 1$,

$$= \frac{\iint\dots\int\left\{x_1x_2\dots x_{n-1}(1-x_1-x_2-\dots-x_{n-1})\right\}^{r-1}dx_1dx_2\dots dx_{n-1}}{\iint\dots\int dx_1dx_2\dots dx_{n-1}}$$

(the integrals extending over all positive values consistent with condition $x_1+x_3+\dots+x_{n-1} \not> 1$)

$$= \frac{\left\{\Gamma(r)\right\}^{n-1}}{\Gamma\left\{r(n-1)\right\}}\int_0^1(1-h)^{r-1}h^{r(n-1)-1}dh \Big/ \frac{\left\{\Gamma(1)\right\}^{n-1}}{\Gamma(n)}$$

$$= \Gamma(n)\frac{\left\{\Gamma(r)\right\}^{n-1}}{\Gamma\left\{r(n-1)\right\}}\frac{\Gamma(r)\Gamma\left\{r(n-1)\right\}}{\Gamma\left\{r(n-1)+r\right\}} = \Gamma(n)\left\{\Gamma(r)\right\}^n \div \Gamma(nr).$$

Putting $r = \frac{3}{2}$, we get the mean value of

$$(x_1x_2x_3\dots x_n)^{\frac{1}{2}} = \Gamma(n)\left\{\Gamma(\tfrac{3}{2})\right\}^n \div \Gamma(n\tfrac{3}{2}).$$

11659. (Professor ZERR.)—From an unknown number of balls, each equally likely to be red, white, or blue, $m+n+p$ are drawn out, and m turn out red, n white, p blue. If $r+s+t$ more are drawn, find the chance that r are red, s white, and t blue.

Solution by H. J. WOODALL, A.R.C.S.

Let us represent the idea of m red balls, n white balls, and p blue balls by (m, n, p); $m>n>p$ say.

If there are (a, a, a) balls, then the probability of drawing (m, n, p) is

$$\left[\{a\,!\}^3\,(3a-m-n-p)\,!\right]/\left[3a\,!\,(a-m)\,!\,(a-n)\,!\,(a-p)\,!\right] = \mathrm{A}a.$$

But the drawing happened to be (m, n, p), therefore the probability of this cause $= \mathrm{A}a/\Sigma\mathrm{A}$ (between $a = m$ and $a = \infty$).

If now $(r+s+t)$ balls are drawn from the remaining $3a-m-n-p$ balls, the probability that, coming from (a, a, a), they are (r, s, t) is

$$\frac{(a-m)\,!\,(a-n)\,!\,(a-p)\,!\,(3a-m-n-p-r-s-t)\,!}{(3a-m-n-p)\,!\,(a-m-r)\,!\,(a-n-s)\,!\,(a-p-t)\,!} = \mathrm{B}a.$$

Required probability $=$ sum of probabilities.

$$= \Sigma\left[\{\mathrm{A}a/\Sigma\mathrm{A}\}\,\mathrm{B}a\right] = \Sigma\mathrm{A}a\mathrm{B}a/\Sigma\mathrm{A}a$$

$$= \frac{\Sigma\,\dfrac{\{a\,!\}^3\,(3a-m-n-p-r-s-t)\,!}{(3a)\,!\,(a-m-r)\,!\,(a-n-s)\,!\,(a-p-t)\,!}}{\Sigma\,\dfrac{\{a\,!\}^3\,(3a-m-n-p)\,!}{(3a)\,!\,(a-m)\,!\,(a-n)\,!\,(a-p)\,!}},$$

the summation being taken between the values m and ∞ of a.

11609. (A. J. PRESSLAND, M.A.)—If s_n be the side of the regular n-gon in a circle, examine the following approximations:—(1) $s_{29} = \frac{1}{8}s_3$, $s_{31} = \frac{1}{7}s_4$, and (2) $s_{10}-s_{17} = s_{25}$, $s_{11}^2 = s_8^2-s_{12}^2$, $s_{24}+s_{34} = s_{14}$.

Solution by H. J. WOODALL, A.R.C.S.

Take circle with radius $= \frac{1}{2}$ unit; then $s_n = \sin \pi/n$.

(1) $s_{29} = \cdot 10812$, $s_3 = \cdot 866025$; $\therefore\ \frac{1}{8}s_3 = \cdot 10825$, $\frac{1}{8}$ per cent. in excess;
$s_{31} = \cdot 10117$, $s_4 = \cdot 707107$; $\therefore\ \frac{1}{7}s_4 = \cdot 101015$, $\frac{1}{6}$ per cent. in defect.

(2) $s_{10} = \cdot 309017$, $s_{17} = \cdot 183750$, $s_{10}-s_{17} = \cdot 125267$, $s_{25} = \cdot 125333$;
$\therefore\ s_{10}-s_{17} = \frac{1}{20}$ per cent. in defect.

$s_{11}^2-s_8^2+s_{12}^2 = -\cdot 00086 =$ about $\frac{1}{10}$ per cent. of s_{11}^2 in excess.

$s_{11}^2 = \cdot 0794$ about.

$s_{24}+s_{34} = \cdot 130526 + \cdot 092268 = \cdot 222794.$

$s_{14} = \cdot 222521$, $\frac{1}{8}$ per cent. in defect.

8566. (W. J. C. SHARP, M.A.)—If P_n denote the number of partitions, without repetitions, of a number n, and Q_n the number into odd parts, prove that (1) $P_n = Q_n + Q_{n-2} P_1 + Q_{n-4} P_2 + \&c.$, and (2) the same formula holds if P_n and Q_n denote the numbers of partitions with repetitions.

Solution by H. J. WOODALL, A.R.C.S.

Assuming the usual notation (see CHRYSTAL's *Algebra*), it is obvious that $$\mathrm{P}u\,(2n \mid * \mid \text{even}) = \mathrm{P}u\,(n \mid * \mid *),$$ and that $$\mathrm{P}\,(2n \mid * \mid \text{even}) = \mathrm{P}\,(n \mid * \mid *).$$

Expanding (1) by means of these symbols, we get

$$\mathrm{P}u\,(n \mid * \mid *) = \mathrm{P}u\,(n \mid * \mid \text{odd}) + \mathrm{P}u\,(n-2 \mid * \mid \text{odd}) \times \mathrm{P}u\,(2 \mid * \mid \text{even})$$
$$\ldots + \mathrm{P}u\,(n-2r \mid * \mid \text{odd}) \times \mathrm{P}u\,(2r \mid * \mid \text{even}) + \&c.$$

And it will be noticed that each of the terms of the right-hand side is different. Also, the partitions of a number (with or without repetitions) may be uniquely divided into terms which include—first, those where all the parts are odd; second, where the sum of the even parts is 2, 4, 6, ... $2r$ ..., and, obviously, the partitions will be different in each case. Hence the theorem follows.

[The question arises whether the theorem can be proved by pure analysis.]

8066. (By SATIS CHANDRA RÂY.)—Sum to infinity the series of which the nth term is $\dfrac{1}{2^n}\left(\dfrac{\sin^{2n}\theta}{n!} - \dfrac{2^n\theta^{2n}}{2n!}\right)$.

Solution by H. W. CURJEL, B.A.

$$\Sigma_1^\infty \frac{1}{2^n}\left\{\frac{\sin^{2n}\theta}{n!} - \frac{2^n\theta^{2n}}{2n!}\right\} = \Sigma_1\left(\frac{\sin^2\theta}{2}\right)^n\frac{1}{n!} - \Sigma_1^\infty\frac{\theta^{2n}}{2n!}$$

$$= e^{\frac{1}{2}(\sin^2\theta)} - 1 - \left\{\frac{e^\theta + e^{-\theta}}{2} - 1\right\} = e^{\frac{1}{2}(\sin^2\theta)} - \frac{e^\theta + e^{-\theta}}{2} = e^{\frac{1}{2}(\sin^2\theta)} - \cosh\theta.$$

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785. (MORTIMER COLLINS.) — The captain of a steamer, carrying dispatches, finds himself 24 miles from the nearest point on the shore, which point is distant 50 miles along the coast from the town he has to reach. The speed of the steamer is 12 miles per hour; an express on shore can be obtained which will travel 15 miles an hour. Where ought he to land his dispatches so as to convey them to their destination in the least possible time?

Solution by Professor G. B. M. Zerr.

Let B be the nearest point on shore from the ship, x = the distance from B to point where he must land. Then $(x^2+24^2)^{\frac{1}{2}}/12+(50-x)/15$ must be a minimum. Taking differential coefficients of the two terms, we find $15x/(x^2+576)^{\frac{1}{2}} = 12$ for the condition; therefore $x = 32$ miles from B; time $= 4\frac{8}{15}$ hrs. = 4 hrs. 32 min.

3797. (Dr. S. H. Wright, M.A.)—A person sends a publisher money, and directs his paper to be sent as long as the money and its interest last. The interest is 7 per cent. *simple* interest, and price of the paper is 1 dol. 50 c. a year in advance. Had *compound* interest been agreed upon, the paper could have been sent one year longer. Find what sum of money was sent, and how long the paper will be continued.

Solution by H. J. Woodall, A.R.C.S.

Assuming that, at simple interest, the successive amounts of interest are not "added to the principal" until that reduced principal becomes too small to bear the annual charge, in which case there will be no further interest, (1) then the successive amounts of interest will be

$$7(x-1\tfrac{1}{2})/100, \quad 7(x-3)/100, \quad \ldots, \quad 7\left\{x-1\tfrac{1}{2}(n-1)\right\}/100.$$

Subscription for the nth year will be made up of reduced principal + sum of these amounts of interest.

$$1\tfrac{1}{2} = x-(n-1)\,1\tfrac{1}{2}+7x(n-1)/100-21(n-1)\,n/100;$$

$$\therefore \quad x(28n+372) = 21n^2+579 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

(2) The sum in the bank varies by the payment of 1·5 dollars at commencement of year and receipt of interest at end of year;

i.e., $$\left[\left\{(x-1\tfrac{1}{2})\,1{\cdot}07-1{\cdot}5\right\}1{\cdot}07-1{\cdot}5\right]\, n \text{ years} = 1\tfrac{1}{2};$$

$$\therefore \quad 14x = 321\left\{1-(1{\cdot}07)^{-(n+1)}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Eliminating x, $-49n^2+147n+19902 = 214(7n+93)(1{\cdot}07)^{-(n+1)}$;

whence $n = 10$, and $x = 12{\cdot}02$ dollars.

11861. (Professor Barisien.)—D'un point fixe P du plan d'une lemniscate de Bernouilli, on mène une sécante quelconque qui rencontre la lemniscate en quatre points A, B, C, D. Le lieu du centre des moyennes distances des quatre points A, B, C, D, lorsque la sécante pivote autour du point P, est un cercle qui reste invariable pour toutes les lemniscates ayant même centre et mêmes directions d'axes.

Solution by Professor Droz Farny; H. W. Curjel, B.A.; *and others.*

Si, dans l'équation $u = 0$ d'une courbe du n^{me} degré, les termes du n^{me}

et du $(n-1)^{me}$ degré sont respectivement représentés par u_n et u_{n-1}, l'équation du diamètre de Newton, conjugué à la direction α par rapport à l'axe des x, sera $$u_{n-1}+x\frac{du_n}{dx}+y\frac{du_n}{dy}=0,$$ si dans u_{n-1} et u_n on a soin de remplacer x et y par $\cos\alpha$ et $\sin\alpha$.

La lemniscate ayant pour équation $(x^2+y^2)^2-a^2(x^2-y^2)=0$, on a

$$u_4=(x^2+y^2)^4,\quad u_3=0,\quad \frac{du_4}{dx}=4(x^2+y^2)x,\quad \frac{du_4}{dy}=4(x^2+y^2)y;$$

d'où, pour l'équation du diamètre, $x\cos\alpha+y\sin\alpha=0$.

Le centre des moyennes distances sur une sécante quelconque est donc le pied de la perpendiculaire abaissée sur cette sécante du centre de la lemniscate. Le lieu cherché sera la circonférence décrite sur OP comme diamètre.

11825. (Professor HUMBERT.)—Montrer que (1) les équations
$$bx^2+2x-b=0,\quad ax^4+4x^3-6ax^2-4x+a=0$$
ne peuvent avoir une racine commune sans en avoir deux; (2) trouver la relation qui doit existre entre a et b pour qu'elles aient deux racines communes; et (3) expliquer les résultats par la trigonométrie.

Solution by H. W. CURJEL, B.A.; Professor MADHAVARAO; *and others.*

(1) It is evident, from the form of the equations, that if α is a root of either equation, $-1/\alpha$ is also a root of the same equation. Hence, if α is a common root, $-1/\alpha$ is also a common root.

(2) Dividing the left-hand side of the second equation by that of the first, and equating the coefficient of x and the absolute term of the remainder each to zero, we get, in both cases, the condition for two common roots, $a=2b/(1-b^2)$.

(3) If we put $x=\tan\theta$, the equations become
$$\cot 2\theta=b,\quad a\tan^2 2\theta-2\tan 2\theta-a=0,$$
each value of $\tan 2\theta$ giving two values of $\tan\theta$. The condition for a common root becomes $\tan\alpha=2\tan\beta/(1-\tan^2\beta)=\tan 2\beta$, where $a=\tan\alpha$, $b=\tan\beta$.

11879. (Rev. T. R. TERRY, M.A.)—Prove the following formulæ without using the methods of the infinitesimal calculus:—

(1) $\frac{1}{4}\pi=\cos x-\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x-\dots$;

(2) $\frac{1}{2}\tanh^{-1}(\sin x)=\sin x-\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x-\dots$.

Solution by W. J. Dobbs, B.A.; Professor Zerr; *and others.*

Let C = first series, S = second series; then

$$C+iS = e^{ix}-\tfrac{1}{3}e^{3ix}+\tfrac{1}{5}e^{5ix}-\dots = -\tfrac{1}{2}i\log\left[(1+ie^{ix})/(1-ie^{ix})\right]$$
$$= -\tfrac{1}{2}i\log\left[i\cos x/(1+\sin x)\right] = -\tfrac{1}{2}i\log i+\tfrac{1}{2}i\log\left[(1+\sin x)/\cos x\right]$$
$$= -\tfrac{1}{2}i\,(2ir\pi+\tfrac{1}{2}i\pi)+\tfrac{1}{2}i\tanh^{-1}(\sin x) \quad (r \text{ being some integer})$$
$$= r\pi+\tfrac{1}{4}\pi+\tfrac{1}{2}i\tanh^{-1}(\sin x).$$

Hence we have $C = r\pi+\frac{1}{4}\pi$, $S=\frac{1}{2}\tanh^{-1}(\sin x)$; and evidently r must $=0$.

11623. (Professor Bhattacharya.) — The sum of two positive quantities is known: prove that it is an even chance that their product will be not less than three-fourths of their greatest possible product.

Solution by H. W. Curjel, B.A.; Professor Zerr; *and others.*

Let the given sum be $2a$, and the two numbers be $(a+x)$, $(a-x)$. Then all values of x between 0 and a are equally likely; hence the chance is even that x is not greater than $\frac{1}{2}a$; *i.e.*, that $(a+x)(a-x)$ is not less than $\frac{3}{4}a^2$, or the greatest possible product of the two numbers.

10409. (Professor Sircom.)—Required a simple method of approximating to the imaginary roots of numerical equations.

Solution by H. J. Woodall, A.R.C.S.

This may be obtained from the equation of squared differences of the roots, which can be got by Professor Malet's method (see Quest. 11255, Vol. lvii., p. 41). Then negative roots correspond to imaginary roots of the original equation. By this means (applied to every such negative root) we obtain the 2β's in the $\alpha\pm i\beta$. Next substitute $\alpha\pm i\beta$ in the original equation, and obtain two equations (by equating real and imaginary parts to zero). Then α will be found as the common factor of these equations.

[Prof. Sircom remarks that Mr. Woodall's process is ingenious, but it does not appear to be much simpler than the usual one of substituting $\alpha+i\beta$ in the given equation, equating real and imaginary parts separately to zero, eliminating α or β, and approximating to the real roots of the resulting equation.]

8076. (By D. EDWARDES.)—One end of a heavy elastic string of natural length l is attached to a fixed point, the string being initially vertical and at its natural length. Prove that its length oscillates between l and $l(1+l/l')$, where l' is the length of a similar string whose weight is the modulus of elasticity.

Solution by Professor ZERR.

If S be the distance at the time t of the section of the string at the distance x from the end which is moved with the acceleration of gravity, T be the tension, μ the modulus of elasticity of the substance, and m the mass per unit of length of the string, the equations of motion are:—

$$m\,(d^2\mathrm{S}/dt^2-g) = d\mathrm{T}/dx,\quad \mathrm{T}/\mu = d\mathrm{S}/dx-1;$$

$$\therefore\quad d^2\mathrm{S}/dt^2-g = l'g\,d^2\mathrm{S}/dx^2,\quad \text{since } l'g = \mu/m;$$

$$\therefore\quad \mathrm{S} = f\left[x+(l'g)^{\frac{1}{2}}t\right]+\phi\left[x-(l'g)^{\frac{1}{2}}t\right]+\tfrac{1}{2}gt^2$$

(1) when $t=0$, $d\mathrm{S}/dt = 0$, $d\mathrm{S}/dx = 1$ for all values of x from 0 to l;

$$\therefore\quad f'(y)-\phi'(y) = 0,\quad f'(y)+\phi'(y) = 1 \quad \text{for values of } y \text{ from 0 to } l;$$

(2) when $x=0$, $d\mathrm{S}/dt = 0$ for all values of t;

$$\therefore\quad f'(y)-\phi'(-y)+y/l' = 0 \text{ for all positive values of } y;$$

(3) when $x=l$, $d\mathrm{S}/dx = 1$ for all values of t;

$$\therefore\quad f'(l+y)+\phi'(l+y) = 1 \quad \text{for all positive values of } y;$$

$$\therefore\quad f''(y) = \phi''(y) \quad \text{for all values of } y \text{ from 0 to } l;$$

and $f''(y)+\phi''(-y)+1/l' = 0$, $f''(l+y) = \phi''(l-y)$ for all positive values of y.

Therefore $\left\{f''(y)\right\}_l^{2l} = \left\{f''(l+y)\right\}_0^l = \left\{\phi''(l-y)\right\}_0^l = 0$;

and, since $\quad f''(2l+y) = \phi''(-y) = -f''(y)-1/l'$,

therefore $\quad \left\{f''(y)\right\}_{2l}^{4l} = -1/l',\quad \left\{f''(y)\right\}_{4l}^{6l} = 0,\quad$ &c.

The relative accelerations of the end of the string

$$= l'gf''\left[l+(l'g)^{\frac{1}{2}}t\right]+l'g\,\phi''\left[l-(l'g)^{\frac{1}{2}}t\right]+g = 2l'gf''\left[l+(l'g)^{\frac{1}{2}}t\right]+g,$$

and from $t=0$ to $t = l/(l'g)^{\frac{1}{2}}$ it is g; from $t = l/(l'g)^{\frac{1}{2}}$ to $t = 3l/(l'g)^{\frac{1}{2}}$ it is $-g$; from $t = 3l/(l'g)^{\frac{1}{2}}$ to $t = 5l/(l'g)^{\frac{1}{2}}$ it is g, and so on.

The length of the string therefore oscillates between $l+l^2/l'$ and l.

10407. (Professor MADHAVARAO.)—Find the values of

$$\int_0^1 \log(1+x)\,(\log x)^2\frac{dx}{x},\quad \int_1^0 \log(1-x)\,(\log x)^2\frac{dx}{x},\quad \int_0^1 \log\left(\frac{1+x}{1-x}\right)(\log x)^2\frac{dx}{x}.$$

Solution by Professor MÁTILÁL MÁLLIK, M.A.

$$(1)\quad \int_0^1 \log(1+x)(\log x)^2\frac{dx}{x} = \int_0^1\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)(\log x)^2\,dx.$$

Hence, by integrating each term separately (see WILLIAMSON's *Integral Calculus*, 5th edition, p. 123), we get the expression equal to

$$2\,!\left\{\frac{1}{1^4}-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\&c.\right\} = 2\,!\,\frac{7\pi^4}{720}$$ (see CASEY's *Trigonometry*, Ex. 38th, p. 232).

$$(2)\quad \int_1^0 \log(1-x)(\log x)^2\frac{dx}{x} = \int_0^1\left(1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\dots\right)(\log x)^2\,dx$$

$$= 2\,!\left\{\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\dots\right\} = 2\,!\times\frac{\pi^4}{90} = \frac{\pi^4}{45}$$ (Ex. 19th, p. 230, CASEY's *Trigonometry*).

$$(3)\quad \int_0^1 \log\left(\frac{1+x}{1-x}\right)(\log x)^2\frac{dx}{x}$$

$$= \int_0^1 \log(1+x)(\log x)^2\frac{dx}{x} + \int_1^0 \log(1-x)(\log x)^2\frac{dx}{x}$$

$$= \frac{7\pi^4}{360}+\frac{\pi^4}{45} = \frac{15\pi^4}{360} = \frac{\pi^4}{24}.$$

10784. (J. J. BARNIVILLE.)—Divide a right angle into 255 equal parts.

Solution by the PROPOSER.

This problem is solved by inscribing polygons of 15 and 17 sides. See AMPÈRE's solution for the latter problem in CATALAN's volume.

11380. (Professor BERNÈS.) — Si, sur deux droites anti-parallèles relativement à l'angle A du triangle ABC et issues du sommet A, on considère deux couples de points inverses M et M′, N et N′, le second point d'intersection des circonférences AMN′, ANM′ est sur la circonférence ABC.

Solution by W. J. GREENSTREET, M.A.

Let AM. AM′ cut the circumcircle of ABC in D, D′, and suppose N lies on AM, &c. Then we have

$(\mathrm{MND}\,\infty) = (\mathrm{M'N'}\,\infty\,\mathrm{D}) = (\mathrm{N'M'D'}\,\infty)$, or $\mathrm{DM/DN} = \mathrm{D'N'/D'M'}$;

i.e., the circles ADD′, AMN′, AM′N have a common point.

10401. (W. J. GREENSTREET, M.A.)—Prove that a train in motion meets with greater resistance from a cross-wind than from a head-wind.

Solution by Professor ZERR.

Let A = the area exposed at right angles to the wind; F = the force of the wind in pounds; V = the velocity of train per second; v = velocity of plane A in the direction of the wind, + when it moves opposite, 0 when it moves at right angles, and − when it moves with the wind.

Then $F = 0{\cdot}002288A\,(V \pm v)^2$, generally;

$= 0{\cdot}002288A\,(V+v)^2$, for head-wind:

$= 0{\cdot}002288AV^2$, for cross-wind at right angles;

$= 0{\cdot}002288A \sin\theta\,(V \pm v\cos\theta)^2$, for a cross-wind at an angle θ with direction of train;

but A is so much larger for a cross-wind than for a head-wind that F is greater for the former.

10601. (Professor LEROUX).—Eliminer x, y, z entre les équations

$$m\sin x + n\cos x = m\sin y + n\cos y = 1,$$

$$\frac{\sin x}{\sin z} + \frac{\cos x}{\cos z} = \frac{\sin y}{\sin z} + \frac{\cos y}{\cos z} = -1.$$

Solution by W. J. GREENSTREET, M.A.; H. J. WOODALL; *and others.*

$m\sin x + n\cos x - 1 = 0 \;\ldots\; (\alpha)$; $\quad \sin x/\sin z + \cos x/\cos z + 1 = 0 \;\ldots\; (\gamma)$;

$m\sin y + n\cos y - 1 = 0 \;\ldots\; (\beta)$; $\quad \sin y/\sin z + \cos y/\cos z + 1 = 0 \;\ldots\; (\delta)$.

From (α) and (β),

$$m/(\cos y - \cos x) = n/(\sin x - \sin y) = +1/\sin(x-y);$$

$\therefore\; m = +\sin\tfrac{1}{2}(x+y)/\cos\tfrac{1}{2}(x-y)$, and $n = +\cos\tfrac{1}{2}(x+y)/\cos\tfrac{1}{2}(x-y)$.

From (γ) and (δ), in the same way,

$\operatorname{cosec} z = -\sin\tfrac{1}{2}(x+y)/\cos\tfrac{1}{2}(x-y)$, and $\sec z = -\cos\tfrac{1}{2}(x+y)/\cos\tfrac{1}{2}(x-y)$;

whence we have, directly, $m^2 + n^2 = m^2n^2$.

11228. (Professor ZERR.)—Required the average volume of (1) the sphere; (2) the ellipsoid described upon that part of the major axis upon which the ellipse is described whose average area is required in Quest. 10836.

Solution by the PROPOSER.

With the same figure and same notation as I used in solution of Quest. 10836, since volume of sphere is $\frac{4}{3}\pi v^3$, and of the ellipsoid $\frac{4}{3}\pi v z^2$, we get

$$(1)\quad \Delta = \frac{\int_0^a \left[\int_0^{x_1}\int_0^x \frac{4}{3}\pi v^3\,dx\,dy + \int_{x_1}^a\int_0^{y_1} \frac{4}{3}\pi v^3\,dx\,dy\right] dw}{\int_0^a \left[\int_0^{x_1}\int_0^x dx\,dy + \int_{x_1}^a\int_0^{y_1} dx\,dy\right] dw}$$

$$= \frac{36}{a^3(3\pi+2)}\int_0^a \left[\int_0^{x_1}\int_0^x \frac{4}{3}\pi v^3\,dx\,dy + \int_{x_1}^a\int_0^{y_1} \frac{4}{3}\pi v^3\,dx\,dy\right] dw$$

$$= \frac{\pi a^3}{2520}\left(\frac{2205\pi+2012}{3\pi+2}\right);$$

$$(2)\quad \Delta' = \frac{\int_0^a \left[\int_0^{x_1}\int_0^x\int_0^v \frac{4}{3}\pi v z^2\,dx\,dy\,dz + \int_{x_1}^a\int_0^{y_1}\int_0^v \frac{4}{3}\pi v z^2\,dx\,dy\,dz\right] dw}{\int_0^a \left[\int_0^{x_1}\int_0^x\int_0^v dx\,dy\,dz + \int_{x_1}^a\int_0^{y_1}\int_0^v dx\,dy\,dz\right] dw}$$

$$= \frac{360}{a^4(15\pi+17)}\int_0^a \left[\int_0^{x_1}\int_0^x\int_0^v \frac{4}{3}\pi v z^2\,dx\,dy\,dz + \int_{x_1}^a\int_0^{y_1}\int_0^v \frac{4}{3}\pi v z^2\,dx\,dy\,dz\right] dw$$

$$= \frac{\pi a^3}{33264}\left(\frac{72765\pi+447298}{15\pi+17}\right).$$

11257. (Professor SHIELDS, M.A.)—The extreme point of the minute-hand of a clock circumscribes a circle, and a house-fly A, starting at a certain point, and travelling around the circle at the rate of 25 inches per minute, reaches the same starting point at 55 minutes past 1 o'clock p.m., while another fly B, starting at the *same point* 10 seconds later, and travelling around the circle at the rate of 24 inches per minute, reaches the same starting point $55\frac{4}{15}$ minutes past 1 o'clock the same day. Give (1) the starting time of each fly; (2) the length of the minute-hand, and (3) the area of the clock's face.

Solution by W. J. GREENSTREET, M.A.; Professor ZERR; *and others.*

If the fly A starts at x' past one, and r be the length of the minute-hand, we have $x + 2\pi r/25 = 55$; $x + \frac{1}{6} + 2\pi r/24 = 55\frac{4}{15}$;

$$\therefore\ x = 52'\ 36''\ \text{past one};\quad \therefore\ \text{B starts at } 52'\ 46'' \ldots.$$

$$r = 7\tfrac{7}{11}\ \text{in., and area} = 176\tfrac{10}{11}\ \text{sq. in.}$$

11979. (H. J. Woodall, A.R.C.S.)—Show how to inscribe a square in a given quadrilateral.

Solution by R. F. Davis, M.A.

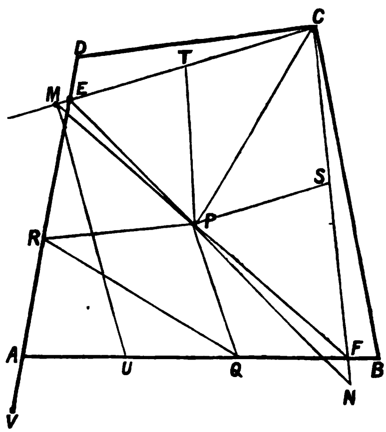
Let ABCD be the given quadrilateral. Through C draw CE perpendicular to CB, meeting AD in E; and CF perpendicular to CD, meeting AB in F. Take a point U on AB such that, if MU be perpendicular to CE, MU = MC (*i.e.*, by making the angle UCE = 45°).

Similarly, take a point V on AD such that, if NV be perpendicular to CF,

NV = NC.

Then the point P of intersection of EN, FM determines the solution.

For, drawing PQ, PR, PS, PT parallel to CB, CD, CE, CF respectively, CSPT is a parallelogram; and



$$PQ : MU = FP : FM = PS : MC.$$

But MU = MC; hence PQ = PS. Similarly, PR = PT = SC.

Then the triangles PSC, QPR have two sides of the one respectively equal and perpendicular to the two corresponding sides of the other. Therefore CP is also equal and perpendicular to QR; and the square described on QR has its two other angular points lying on CB, CD respectively.

11930. (Professor B. O. Peirce.)—Each of a number of equal heavy particles, which make up a system S, is constrained to remain on one of a set (P) of parallel straight lines in a plane, equally spaced at a distance a apart. Between the members of each pair of adjacent particles there acts an attractive force proportional to the distance, and of absolute value μ. There are $n+2$ consecutive particles in the system, and the extreme particles are fixed at two points A and B in a line which cuts P at right angles. The system S′ is in all respects similar to S, except that it consists of $n+1$ particles and the extreme particles are free to move on their lines. However the particles in S and S′ may be moving under the action of the internal forces, it is evident that the configuration of each of the systems (provided the centre of gravity of S′ is at rest) may be completely stated by a combination of n simple harmonic terms of definite periods. Show that $\delta_{n+1} \div \Delta_n = D^2$, where D represents differ-

entiation with respect to the time, Δ_n represents the determinant of n rows and n columns:—

$$\left|\begin{array}{cccccc} D^2+2k & -k & 0 & 0 & & \\ -k & D^2+2k & -k & 0 & & \\ 0 & -k & D^2+2k & -k & & \\ 0 & 0 & -k & D^2+2k & & \\ & & & & D^2+2k & -k \\ & & & & -k & D^2+2k \end{array}\right|$$

and δ_n represents a determinant with all its elements identical with those of Δ_n except the first and last elements of the principal diagonal, which are D^2+k instead of D^2+2k. Hence, prove that the periods of the harmonic terms which are involved in the motion of S′ are the same as those of the terms which are involved in the motion of S.

Solution by the PROPOSER.

Let y_p be the distance of the pth particle of mass m from the line AB in the system S, and from a fixed line perpendicular to the paths in S′. Let $\mu = km$. The equations of motion of the first and last particle in S′ are $\ddot{y}_1 = k(y_2-y_1)$ and $\ddot{y}_{n+1} = k(y_n-y_{n+1})$. The equation of motion of any other particle of either system is $\ddot{y}_p = k(y_{p+1}+y_{p-1}-2y_p)$. Hence, in system S, $\Delta_n(y_p) = 0$; and in system S′, $\delta_{n+1}(y_p) = 0$.

Let δ'_n be a determinant identical with δ_n except that the first element of the principal diagonal is D^2+2k instead of D^2+k.

$\delta'_n = \Delta_n - k\Delta_{n-1}$, $\delta'_{n-1} = \Delta_{n-1} - k\Delta_{n-2}$, and if these values be substituted in $\delta_{n+1} = (D+2k)\,\delta'_n - k^2\delta'_{n-1}$, we get

$$\delta_{n+1} = D^2 . \Delta_n + k\left[\Delta_n - (D^2+2k)\,\Delta_{n-1} + k^2\Delta_{n-2}\right];$$

but $\quad \Delta_n = (D^2+2k)\,\Delta_{n-1} - k^2\,\Delta_{n-2}; \qquad \therefore \quad \delta_{n+1} = D^2 . \Delta_n.$

Besides the factor D^2, which corresponds to a motion of translation of S′ as a whole, δ_{n+1} has the same factors, each of the form $D^2+\alpha^2$, that Δ_n has, and the α's are the periods of the harmonic terms which define the motion.

11966. (Professor MACFARLANE.)—Prove that

$$\frac{\pi^2}{8}\left\{1-2y\,\frac{e^{1/y}-1}{e^{1/y}+1}\right\} = \frac{1}{1^2}\,\frac{1}{1+(\pi y)^2} + \frac{1}{3^2}\,\frac{1}{1+(3\pi y)^2} + \frac{1}{5^2}\,\frac{1}{1+(5\pi y)^2} + \dots .$$

Solution by the PROPOSER.

Let $1/y = x$; then we have

$$\frac{\pi^2}{8}\left\{1-\frac{2}{x}\,\frac{e^x-1}{e^x+1}\right\} = \frac{1}{1^2}\,\frac{1}{1+(\pi/x)^2} + \frac{1}{3^2}\,\frac{1}{1+(3\pi/x)^2} + \dots$$

$$= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots - \left\{\frac{(\pi/x)^2}{1+(\pi/x)^2} + \frac{1}{3^2}\,\frac{(3\pi/x)^2}{1+(3\pi/x)^2} + \dots\right\};$$

therefore, since $$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots,$$

we have $$\frac{\pi^2}{4x}\frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}} = \frac{1}{1+(x/\pi)^2} + \frac{1}{3^2}\frac{1}{1+(x/3\pi)^2} + \dots,$$

This is the same as

$$\frac{1}{2}\frac{\sinh\frac{1}{2}x}{\cosh\frac{1}{2}x} = \left\{\frac{1}{1+x/\pi)^2} + \frac{1}{3^2}\frac{1}{1+(x/3\pi)^2} + \dots\right\}\frac{2x}{\pi^2};$$

therefore, by integration,

$$\log\cosh\tfrac{1}{2}x = \log\left(1+\frac{x^2}{\pi^2}\right) + \log\left(1+\frac{x^2}{3^2\pi^2}\right) + \log\left(1+\frac{x^2}{5^2\pi^2}\right) + \dots;$$

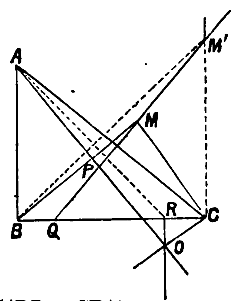
therefore $$\cosh\tfrac{1}{2}x = \left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{3^2\pi^2}\right)\left(1+\frac{x^2}{5^2\pi^2}\right)\dots,$$

which is the infinite product for the hyperbolic cosine, and the analogue of

$$\cos\tfrac{1}{2}x = \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\left(1-\frac{x^2}{5^2\pi^2}\right)\dots.$$

11969. (Professor Mannheim.)—On donne un triangle ABC rectangle en B. On joint les points B et C à un point M du plan du triangle. De A on abaisse la perpendiculaire AP sur BM; en C on élève la perpendiculaire CO à CM. Ces droites se coupent en O et l'on projette ce point orthogonalement en R sur BC. La perpendiculaire MQ sur AC coupe BC au point isotomique de R.

Solution by Professor Droz Farny.



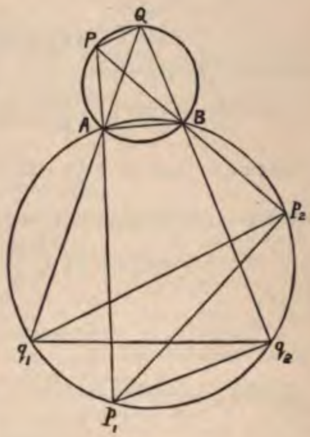
Supposons la droite MQ fixe et le point M mobile sur cette droite. Les droites BM et CM décrivent deux faisceaux perspectifs, et par conséquent les droites AP et CO qui leur sont respectivement perpendiculaires deux faisceaux projectifs. Lorsque M est à l'infini sur QM les rayons AP et CO coïncident avec AC; lorsque M coïncide avec Q les deux rayons AP et CO sont perpendiculaires sur BC; le lieu de O est donc une ligne droite perpendiculaire sur BC. Son point d'intersection R avec BC sera obtenu lorsque M se trouve en M′ intersection de QM avec CM′ perpendiculaire à BC.

On a alors triangles M′CQ, M′BC semblables à CAB, ABR; donc M′C/CQ = CB/AB, M′C/BR = CB/AB.

Il résulte de ces deux proportions CQ = BR, ce qui démontre le théorême de M. Mannheim.

11946. (W. J. Dobbs, B.A.)—Two conics, I. and II., are drawn, each passing through four points, A, B, C, D. P and Q are any two points on I., and PA, PB, QA, QB meet II in the points p_1, p_2, q_1, q_2, respectively. Prove that PQ, q_2p_1, p_2q_1, CD are concurrent; also that p_1p_2, CD, and the tangent at P are concurrent; also that q_1q_2, CD, and the tangent at Q are concurrent.

Solution by H. W. Curjel, B.A.; Professor Droz Farny; *and others.*



Project CD into the circules, then the conics become circles, and CD the line at infinity.

In the projected figure

$$\angle q_2p_1A = \angle ABQ$$

$$= \text{the supplement of } \angle p_1PQ;$$

$\therefore$ p_1q_2 is parallel to PQ; similarly, q_1p_2 is parallel to PQ; therefore p_1q_2, PQ, q_1p_2 are parallel; *i.e.*, p_1q_2, PQ, q_1p_2, CD are concurrent.

If Q moves up to P, PQ becomes the tangent at P and p_1q_2 coincides with p_1p_2; $\therefore$ p_1p_2, CD and the tangent at P are concurrent; similarly, q_1q_2, CD, and the tangent at Q are concurrent.

11943. (R. Tucker, M.A.)—The distances OA, OB, OC of a point O from the angles of an equilateral triangle are a, b, c; if a triangle can be formed with these lengths as sides, prove that, with usual notation, if x is a side of the equilateral triangle, then $x^2 = 2\lambda\cos(60°\pm\omega)$.

Solution by Professors Farny, Bhattacharya, *and others.*

Si le problême est possible, ce qui a lieu lorsque le point O ne se trouve ni sur la circonférence circonscrite ni sur les côtés du triangle, on a en utilisant la formule du Quest. 11869,

$$x^4 - \kappa x^2 + \lambda^2 - 16\Delta^2 = 0,$$

$$\Delta = \text{triangle formé par } a,\ b,\ c, \quad \text{et} \quad \kappa^2 - 4\lambda^2 = -16\Delta^2,$$

donc $$x^2 = \tfrac{1}{2}(\kappa \pm 4\Delta\sqrt{3}).$$

Mais $2\lambda\cos(60\pm\omega) = \lambda(\cos\omega \mp \sqrt{3}\sin\omega) = \tfrac{1}{2}(\kappa \mp 4\sqrt{3}\Delta) = x^2.$

[Mr. Tucker remarks that this question was proposed in the Trigonometry paper of the Senate House (Cambridge) examination for the year 1848, and that it is worked out in the "Solutions" by Messrs. Ferrers

and Jackson (p. 20), where it is shown that

$$x^2 = \tfrac{1}{3}(a^2+b^2+c^2) \pm \tfrac{1}{3}\sqrt{3}\left\{2(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4)\right\}^{\frac{1}{2}}$$

$$= \tfrac{1}{3}\kappa \pm \tfrac{1}{3}\sqrt{3}\left\{(2\lambda^2-\nu^4)\right\}^{\frac{1}{2}}$$

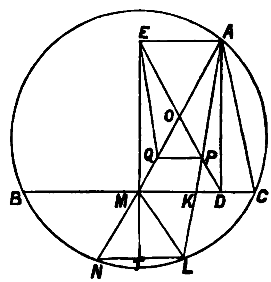
in the notation of his "T.R." circle paper (*Q. J.*, vol. xix., p. 344).

Now $\cos 2\omega = \nu^4/2\lambda^2$, and $\cos\omega = \kappa/2\lambda$;

$$\therefore\ x^2 = 2\lambda\left(\tfrac{1}{2}\cos\omega \pm \tfrac{1}{2}\sqrt{3}\sin\omega\right) = 2\lambda\cos(60^\circ \pm \omega).]$$

11587. (Professor Sollertinsky.)—Soient D, E les projections du sommet A d'un triangle ABC sur le côté BC et sur la médiatrice ME de ce côté. Démontrer que la droite DE passe par le sommet A_2 du second triangle de Brocard, et que $EA_2 = (AB^2 + AC^2)/4AM$.

Solution by W. J. Greenstreet, M.A.



Let the median AM meet the circumcircle in N, and the symmedian AK meet the diagonal ED in P and the circumcircle in L. Then,

as $\angle BBM = \angle CAL$, $\widehat{BN} = CL$,

and, by a well-known problem, NL is parallel to BC.

Hence EM, perpendicular to and bisecting BC, performs the same office for NL; so that

$\angle TML = \angle NMT = \angle AME = \angle MEO$

or ED is parallel to ML.

But O is the mid-point of AM; therefore P is the mid-point, the symmedian AL; and therefore P coincides with A_2.

If PQ be parallel to EA, we see that PE = AQ.

But, as P is mid-point of AL, and PQ parallel to NL, Q is mid-point of AN.

$$\text{Hence } PE = \tfrac{1}{2}AN = \tfrac{1}{2}AN \cdot AM/AM = \tfrac{1}{2}(AM+MN)\,AM/AM$$
$$= \tfrac{1}{2}(AM^2 + AM \cdot MN)/AM = \tfrac{1}{2}(AM^2 + BM \cdot MC)/AM$$
$$= 2(AM^2 + BM^2)/4AM.$$

But $BA^2 + AC^2 = 2(AM^2 + 2BM^2)$; $\therefore\ PE = (BA^2 + AC^2)/4AM.$

11451. (Professor Sollertinsky.)—Sur les côtés d'un angle fixe XAY on prend des longueurs variables AB, AC telles que

$$1/(AB)^2 + 1/(AC)^2 = 2/p^2.$$

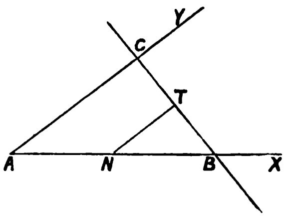
Démontrer que le côté BC enveloppe une ellipse dont les diamètres

conjugués égaux sont dirigés suivant AX, AY ; les demi-axes de cette ellipse sont égaux à $p\cos\frac{1}{2}A$, $p\sin\frac{1}{2}A$. La droite BC touche son enveloppe au pied K de la symédiane AK du triangle ABC.

Solution by J. H. GRACE; Professor SARKAR, M.A.; *and others.*

Taking XA and AY as axes, the tangential equation of the envelope of BC is $l^2+m^2 = 2/p^2$, and therefore the equation of the envelope is $x^2+y^2 = \frac{1}{2}p^2$, which proves that the envelope is an ellipse having AX, AY for equi-conjugates. Each equi-conjugate $= p/\sqrt{2}$. If then a and b are the semi-axes, we have

$$a^2+b^2 = p^2,$$



and $ab = \frac{1}{2}p^2 \sin A = p^2 \sin\frac{1}{2}A \cos\frac{1}{2}A$;

$\therefore\ a = p\sin\frac{1}{2}A$ and $b = p\cos\frac{1}{2}A$,

or *vice versâ*.

Thus the semi-axes are equal to $p\sin\frac{1}{2}A$ and $p\cos\frac{1}{2}A$.

Let T be the point of contact of BC with the envelope. Draw TN parallel to AC. Then $TN.AC = AN.AB = \frac{1}{2}p^2$;

$$\therefore\ TN/AN = AB/AC.$$

But $BN/NT = AB/BC$; $\therefore\ BN/AN = AB^2/AC^2$ or $BT/CT = AB^2/AC^2$, which proves that AT is the symmedian through A.

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800. (MORTIMER COLLINS.)—Solve the functional equations

$$\phi(\psi x) = 16x, \quad \psi(\phi x) = x^4.$$

Solution by E. WHITE, B.A.

We have $\phi\left[\psi\left\{\phi(x)\right\}\right] = 16\phi(x)$ or $\phi(x^4) = 16\phi(x)$.

Let $x = u_t,\quad x^4 = u_{t+1},\quad \phi(x) = v_t,\quad \phi(x^4) = v_{t+1}$;

then $v_{t+1} = 16v_t$, $u_{t+1} = u_t^4$; hence, eliminating t between the solutions of these, $\phi(x) = a(\log x)^2$; thus we have

$$\phi^{-1}(x) = \exp(x/a)^{\frac{1}{2}},\quad \psi(x) = \left\{\phi^{-1}(x)\right\}^4 = \exp 4\,(x/a)^{\frac{1}{2}}.$$

11933. (Professor NEUBERG.)—Soient A′, B′, C′, D′ les projections des sommets d'un tétraèdre ABCD sur un plan quelconque P, et soient A_1, B_1, C_1, D_1 les orthocentres des triangles B′C′D′, C′D′A′, D′A′B′, A′B′C′. Démontrer (1) que les perpendiculaires menées des points A′ et A_1 sur le plan BCD, de B′ et B_1 sur CDA, de C′ et C_1 sur DAB, de D′ et D_1 sur ABC sont sur un même hyperboloïde; (2) que les perpendiculaires

abaissées des milieux des droites $A'A_1$, $B'B_1$, $C'C_1$, $D'D_1$ respectivement sur les plans ACD, CDA, DAB, ABC, concourent en un même point (centre de l'hyperboloïde).

Solution by Professor DROZ FARNY.

Représentons par a', a_1, α les perpendiculaires abaissées de A', A_1 et du milieu de $A'A_1$; de même pour les droites analogues.

Le plan mené par b' et $B'A_1$ étant perpendiculaire sur ACD et CDC'D' est perpendiculaire sur leur ligne d'intersection CD, et par conséquent sur BCD; de même les plans menés par c' et $C'A_1$ et par d' et $D'A_1$ sont perpendiculaires sur BCD; ces trois plans admettent donc une ligne d'intersection commune perpendiculaire à BCD, et qui n'est rien d'autre que a_1; a_1 rencontre donc les perpendiculaires b', c', d' en des points finis et sa parallèle a' à l'infini. On démontrerait de même que b_1, c_1, d_1 rencontrent ces 4 droites; $a'b'c'd'$, $a_1b_1c_1d_1$ sont donc 8 génératrices d'un même hyperboloïde; les 4 premières appartenant au même système, les 4 dernières au second système. On sait que si dans un hyperboloïde l et g sont 2 génératrices parallèles, la droite située à égale distance des deux passe par le centre de l'hyperboloïde; or a' et a_1 sont parallèles et α est située dans le même plan et à égale distance des deux donc α et de même β, γ, δ se croisent au centre de l'hyperboloïde.

11999. (J. BLATER.)—The first (I) of n persons gives to each of the remaining $n-1$ persons x times as much as each already possesses. Then II (the second person) gives in the same manner to each of the remaining $n-1$ persons. III, IV, ... (and finally the last person) give likewise in order, so that at last each has given to all the others the x^{th} multiple of what each possesses at the moment. When this has been completed, each person has an equal share. Example: If there are five persons, and $x = 1$, the initial shares may be respectively 81, 41, 21, 11, 6 pounds, and the final share of each 32 pounds. Required general formulæ for the initial and final shares.

Solution by the PROPOSER; T. SAVAGE, M.A.; *and others.*

The general formula is

$$x\left\{(x+1)^{n-1}+(x+1)^{n-2}\dots(x+1)^2+(x+1)+1\right\}+1 = (x+1)^n,$$

from which we may derive the following forms:—

I, $anx(x+1)^{n-1}+a$;	$n-2$, $anx(x+1)^2+a$;
II, $anx(x+1)^{n-2}+a$;	$n-1$, $anx(x+1)+a$;
... ...	n, $anx+a$.

The sum of the n forms is $an(x+1)^n$. Here a may be an arbitrary number. To avoid a fraction, only integers are to be substituted for a. Further, if x is less than 1, and only integers are admitted, then for a only $(1/x)^n$ or a multiple of $(1/x)^n$ is to be taken.

Let $a = 1$, $n = 5$, $x = 1$;

$$\text{I} = anx(x+1)^4 + a = 81,$$
$$\text{II} = anx(x+1)^3 + a = 41,$$
$$\text{III} = anx(x+1)^2 + a = 21,$$
$$\text{IV} = anx(x+1) + a = 11,$$
$$\text{V} = anx + a = 6.$$

I	II	III	IV	V
81	41	21	11	6
−79	41	21	11	6
2	82	42	22	12
2	−78	42	22	12
4	4	84	44	24
4	4	−76	44	24
8	8	8	88	48
8	8	8	−72	48
16	16	16	16	96
16	16	16	16	−64
32	32	32	32	32

Other Examples.—(2) $n = 2$, $x = 0,01$, $a = 10000$. Initial shares, I = 124, II = 88, III = 64, IV = 48. Final share = 81.

(3) $n = 4$, $x = \frac{1}{10}$, $a = 10000$. Initial shares, I = 15324, II = 14840, III = 14400, IV = 14000. Final share = 14641.

(4) $n = 11$, $x = 1$, $a = 1$. Initial shares, 11265, 5633, 2517, 1409, 705, 353, 177, 89, 45, 23, 12. Final share, 2048.

(5) $n = 11$, $x = 2$, $a = 1$. Initial shares, 1299079, 433027, 144343, 48115, 16039, 5347, 1783, 595, 199, 67, 23. Final share, 177147.

[Using umbral notation, Mr. Woodall gives for the initial shares, (1.1), (2.1), ... $(n.1)$; sum $= s$.

$$(1.2) = (1.1) - x\{(2.1)+(3.1)+\ldots\} = (1.1)(1+x) - xs;$$
$$(2.2) = (x+1)(2.1), \text{ \&c.};$$
$$(1.3) = (x+1)(1.2) = (x+1)\{(x+1)(1.1) - xs\};$$
$$(2.3) = (2.2) - x\{s - (2.2)\} = (x+1)(2.2) - xs;$$

whence finally he finds the $(n+1)^{\text{th}}$ shares to be thus (since they are equal)

$$s/n = (x+1)^{n-1}\{(x+1)(1.1) - xs\} = (x+1)^{n-2}\{x+1)^2(2.1) - xs\} = \text{\&c.};$$
$$(k.n+1) = (x+1)^{n-k}\{(x+1)^k(k, 1) - xs\};$$

and the sum of these last shares

$$= (x+1)^n\{(1.1)+(2.1)+\ldots\} - xs\{(x+1)^{n-1} + (x+1)^{n-2} + \ldots + 1\}$$
$$= s(x+1)^n - xs\{(x+1)^n - 1\}/\{(x+1) - 1\}$$
$$= s(x+1)^n - s\{(x+1)^n - 1\} = s.$$

Since $s/n = (x+1)^{n-k}\{(x+1)^k(k, 1) - xs\}$, we may easily find

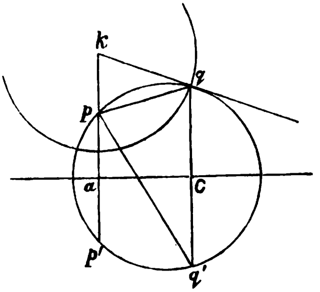
$$(k, 1) = \{s/n + xs(x+1)^{n-k}\}/(x+1)^n.]$$

Mr. Curjel puts the rth person's initial share $= y_r$, and the total shares $= nz$; then each final share $= z$; and the rth person receives $r-1$ times, next gives to each of the others, and lastly receives $n-r$ times; hence his final share

$$z = \{y_r(x+1)^{r-1} - x[nz - y_2(x+1)^{r-1}]\}(x+1)^{n-r}$$
$$= y_r(x+1)^n - x\,nz\,(x+1)^{n-r};$$
$$\therefore \quad y_r = z\{1 + xn(x+1)^{n-r}\}/(x+1)^n.]$$

11885. (I. Arnold.)—A point p and a right line being given in position, find the locus of another point q, so that pq^2 shall be equal to the rectangle under the perpendicular qc (on the given line) and a given line m.

Solution by W. J. Dobbs, B.A.; Professor Zerr; *and others.*



Draw pa perpendicular to the given line, and produce it to p', making $pa = ap'$. Also produce ap to k, making $pk = \frac{1}{2}m$. About $pp'q$ describe a circle cutting qc produced in q', so that evidently $cq' = qc$; then

$$pq^2 = qq' \cdot pk.$$

$\therefore$ triangles kpq, pqq' are similar:

$\therefore$ kq is tangent at q;

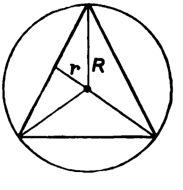
$\therefore$ $kq^2 = kp \cdot kp'$, a constant.

But k is a fixed point; therefore locus required is a circle, centre k.

[Analytically, the solution is obvious.]

11958. (Professor Orchard, M.A., B.Sc.)—In a circle of unit radius is inscribed an equilateral triangle; in this triangle is inscribed a circle, and in this circle another equilateral triangle, and so on—equilateral triangles and circles being inscribed alternately one within another. Prove that the sum of the areas of all the circles is $\frac{4}{3}\pi$ square units.

Solution by C. Morgan, M.A.; T. Savage; *and others.*



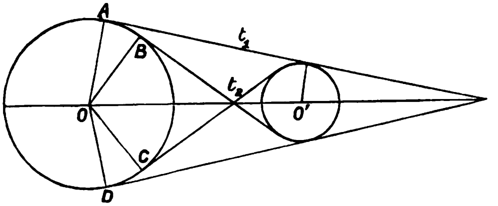
Since $2r = R = 1$, the sum of the areas of all the circles is

$$\pi\left\{1^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{4})^2 \dots \text{to infinity}\right\} = \tfrac{4}{3}\pi \text{ square units.}$$

11928. (Professor Curtis, M.A.)—The lengths of the common tangents to two circles are t_1 and t_2; show that the anharmonic ratio of the points in which they cut any other tangent is $(t_1 - t_2)/(t_1 + t_2)$.

Solution by the PROPOSER.

Let $OO' = D$, the two radii be r_1 and r_2, $AOO' = \alpha$, and $BOO' = \beta$.



Then the ratio is that of

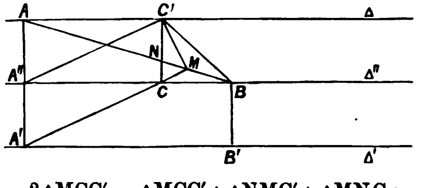
$$(ABCD) = \frac{\sin^2 \frac{1}{2}(\alpha-\beta)}{\sin^2 \frac{1}{2}(\alpha+\beta)} = \frac{1-\cos(\alpha-\beta)}{1+\cos(\alpha+\beta)}.$$

But $\quad \cos\alpha = \frac{r_1-r_2}{D}.\ \cos\beta = \frac{r_1+r_2}{D}.\ \sin\alpha = \frac{t_1}{D}.\ \sin\beta = \frac{t_2}{D};$

$$\therefore \text{ anharmonic ratio } = \frac{D^2-(r_1^2-r_2^2)-t_1t_2}{D^2-(r_1^2-r_2^2)+t_1t_2} = \frac{t_1^2-t_1t_2}{t_1^2+t_1t_2} = \frac{t_1-t_2}{t_1+t_2}.$$

8162. (By Professor DE LONGCHAMPS.)—Soient Δ, Δ' deux parallèles, et Δ'' la parallèle équidistante; soit aussi AA' une perpendiculaire commune à Δ et à Δ'. Ayant pris un point M, arbitrairement, dans le plan de ces droites, MA et MA' rencontrent Δ'', respectivement, aux points B et C. On projette B en B' sur Δ'; et C en C' sur Δ. Démontrer que les trois points C', M, B' sont en ligne droite.

Solution by H. W. CURJEL, B.A.



Let AA' cut Δ'' in A'', and let AM cut CC' in N. Then $A'A''$, CC', BB' are equal and parallel; therefore $A'A''C'C$, $CC'BB'$ are parallelograms.

$\Delta BMC' + \Delta C'NM = \Delta BC'N$
$\qquad = \Delta ACN;$

$$\Delta CNA + \Delta NMC = \Delta ACM = 2\Delta MCC' = \Delta MCC' + \Delta NMC' + \Delta MNC;$$

$$\therefore\ \Delta MCC' = \Delta BMC'; \qquad \therefore\ C'M \text{ passes through } B'.$$

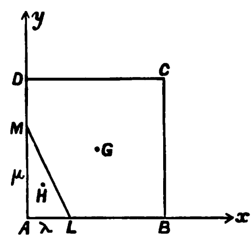
[Projecting the figure, and using the same letters for corresponding points, we obtain the following theorem:—

Let Δ, Δ' be two straight lines, and Δ'', Δ''' (corresponding to line

at ∞) be harmonic conjugates with respect to them. Let any straight line meet Δ, Δ', Δ''' in A, A′, A‴. MA, MA′, (M being any arbitrary point), cut Δ'' in B and C. BA‴, CA‴ cut Δ' and Δ in B′, C′ respectively. Then C′, M, B′ are collinear.]

11714. (J. W. Russell, M.A.)—Four equal similar uniform rods are joined together to form the sides of a square. The square is set floating vertically in a liquid. Show that, if the density of the liquid lies between three and four times that of the rods, then the square can float with one corner only immersed, and with neither diagonal horizontal.

Solution by the Proposer.



The centres of gravity of AL, AM are $(\frac{1}{2}\lambda, 0)$, $(0, \frac{1}{2}\mu)$; therefore H is

$$\frac{\mu^2}{2(\lambda+\mu)}, \quad \frac{\mu^2}{2(\lambda+\mu)},$$

and G is a, a, and GH is perpendicular to LM; hence

$$\left(a - \frac{\mu^2}{2(\lambda+\mu)}\right) \Big/ \left(a - \frac{\lambda^2}{2(\lambda+\mu)}\right) = \frac{\lambda}{\mu}.$$

Also $\qquad \rho(\lambda+\mu) = 8a\sigma,$

or $\qquad \lambda+\mu = 2c$, say.

Hence $\mu a - \mu^3/4c = \lambda a - \lambda^3/4c$; therefore $4ac = \mu^2 + \mu\lambda + \lambda^2$ (rejecting $\lambda = \mu$), $= 4c^2 - \lambda\mu$; therefore λ and μ are given by $t^2 - 2ct + 4c^2 - 4ac = 0$; hence $t = c \pm (4ac - 3c^2)^{\frac{1}{2}}$. Now $4ac > 3c^2$; hence we have $4a > 3c > 12a\sigma/\rho$, $\rho > 3\sigma$. And $c > (4ac-3c^2)^{\frac{1}{2}}$; $\therefore$ $c^2 > 4ac - 3c^2$, $c > a$, $4a\sigma/\rho > a$, $\rho < 4\sigma$.

Also $c + (4ac-3c^2)^{\frac{1}{2}} < 2a$, if $4a^2 - 4ac + c^2 > 4ac - 3c^2$, if $4a^2 - 8ac + 4c^2 > 0$. For $2a > c$, since $2a > \frac{3}{2}c$.

11953. (J. H. Grace.)—Prove that, in any triangle, a focus of the maximum inscribed ellipse is the symmedian point of its pedal triangle.

Solution by Professor Droz Farny.

Soit K le point de Lemoine du triangle ABC, et soient O, O_a, O_b, O_c les centres des cercles circonscrits aux triangles ABC, KBC, KCA, KAB; on sait (Neuberg, Quest. 10622) que K et O sont les foyers de l'ellipse maximale inscrite dans le triangle $O_aO_bO_c$.

Or en représentant par α, β, γ les milieux des droites KA, KB, KC, le triangle $\alpha\beta\gamma$ est le triangle polaire de K dans $O_aO_bO_c$; mais K, étant le centre de similitude des triangles homothétiques ABC et $\alpha\beta\gamma$, est aussi

le point de Lemoine du triangle $\alpha\beta\gamma$. Le triangle $\alpha'\beta'\gamma'$ polaire de O dans le triangle considéré étant évidemment égal et semblablement placé au triangle $\alpha\beta\gamma$, et les points O et K étant homologues dans ces triangles, O est aussi le point de Lemoine de $\alpha'\beta'\gamma'$.

11915. (R. Tucker, M.A.)—ABC, A′B′C′ are two triangles whose sides contain a, b, c; bc, ca, ab units respectively; with the usual notation, show that

$$\cos\omega\cos\omega' = \lambda\kappa/4\lambda'\,; \quad a\sin A' = b\sin B' = c\sin C' = \sqrt{\kappa}\sin\omega'.$$

If, further, A″B″C″ has its sides b^2+c^2, c^2+a^2, a^2+b^2 units, then

$$\cot\tfrac{1}{2}A''.\cot\tfrac{1}{2}B''.\cot\tfrac{1}{2}C'' = \kappa\sqrt{\kappa}/abc, \quad \cot\omega'' = \sin 3\omega.\cos\omega'/\sin\omega,$$

and $$2\Delta' = \Delta''\sin\omega'.$$

Solution by Professors Farny, Zerr, Sarkar, *and others.*

On a $\cos\omega = \dfrac{a^2+b^2+c^2}{2\,(b^2c^2+c^2a^2+a^2b^2)^{\frac{1}{2}}} = \dfrac{\kappa}{2\lambda}$ de même $\cos\omega' = \dfrac{\lambda^2}{2\lambda'}$,

$$\cos\omega.\cos\omega' = \kappa\lambda/4\lambda',$$

$$\frac{bc}{\sin A'} = 2R' = \frac{a^2b^2c^2}{2\Delta'}, \quad a\sin A' = b\sin B' = c\sin C' = \frac{2\Delta'}{abc}.$$

Or $$\sin\omega' = \frac{2\Delta'}{\lambda'} = \frac{2\Delta'}{abc\sqrt{\kappa}}, \quad a\sin A' = \sqrt{\kappa}\sin\omega'.$$

(1) $\cot\frac{1}{2}A''.\cot\frac{1}{2}B''.\cot\frac{1}{2}C'' = p''^2/\Delta'' = (a^2+b^2+c^2)^2/\Delta'' = \kappa^2/\Delta''$.

Or $$\Delta'' = \left\{(a^2+b^2+c^2)\,a^2b^2c^2\right\}^{\frac{1}{2}} = abc\sqrt{\kappa},$$

$$\cot\tfrac{1}{2}A''.\cot\tfrac{1}{2}B''.\cot\tfrac{1}{2}C'' = \kappa\sqrt{\kappa}/abc.$$

(2) $$\cot\omega'' = \frac{(b^2+c^2)^2+(a^2+b^2)^2+(a^2+c^2)^2}{4\Delta''} = \frac{\lambda^2+a^4+b^4+c^4}{2abc\sqrt{\kappa}}.$$

Mais $$\frac{\sin 3\omega.\cos\omega'}{\sin\omega} = (3-4\sin^2\omega)\cos\omega' = \frac{a^4+b^4+c^4+\lambda^2}{\lambda^2}\cdot\frac{\lambda^2}{2\lambda'} = \cot\omega''.$$

(3) Enfin $\Delta'' = abc\sqrt{\kappa} = \lambda'$, $\sin\omega' = \dfrac{2\Delta'}{abc\sqrt{\kappa}}$, donc $\Delta''\sin\omega' = 2\Delta'$.

[Mr. Tucker gives this solution:—For the triangle ABC,

$$\kappa = a^2+b^2+c^2, \quad \lambda^2 = b^2c^2+c^2a^2+a^2b^2, \quad \cot\omega = \kappa/4\Delta\,;$$

hence $\kappa' = \lambda^2$, $\lambda'^2 = a^2b^2c^2\kappa$, $\sec^2\omega' = 4a^2b^2c^2\kappa/\lambda^4$, $\cos\omega = \kappa/2\lambda$;

$$\therefore\quad \cos\omega\cos\omega' = \frac{\kappa}{2\lambda}\cdot\frac{\lambda^2}{2abc\sqrt{\kappa}} = \frac{\sqrt{\kappa}.\lambda}{4abc} = \frac{\lambda\kappa}{4\lambda'}.$$

Again, $2\Delta' = \lambda'\sin\omega'$ and also $= abc\,(a\sin A')$;

$$\therefore\quad a\sin A' = b\sin B' = c\sin C' = \lambda'\sin\omega'/abc = \sqrt{\kappa}.\sin\omega'.$$

Again, $$\cos A'' = \frac{a^2\kappa-b^2c^2}{a^2\kappa+b^2c^2}\,; \quad\therefore\quad \cot^2\tfrac{1}{2}A'' = \frac{a^2\kappa}{b^2c^2}\,;$$

hence $$\cot\tfrac{1}{2}A''.\cot\tfrac{1}{2}B''.\cot\tfrac{1}{2}C'' = \kappa\sqrt{\kappa}/abc = \kappa^2/\lambda'.$$

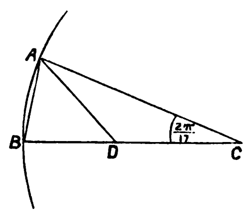
Since $\cot \omega'' = \left\{(a^2+b^2)^2+(b^2+c^2)^2+(c^2+a^2)^2\right\}/4\Delta'' = (\nu^4+\lambda^2)/2\Delta''$,
and $\Delta'' = \lambda'$;

$$\therefore \quad \cot \omega'' = \frac{\kappa^2-\lambda^2}{2\lambda'} = \frac{\kappa^2-\lambda^2}{\lambda^2}\cdot\frac{\kappa'}{2\lambda'} = \frac{\sin 3\omega}{\sin\omega}\cdot\cos\omega',$$

and $\Delta''\sin\omega' = \lambda' \,.\, 2\Delta'/\lambda' = 2\Delta'$.]

11541. (A. J. Pressland, M.A.)—Examine the following approximations to the side of the regular inscribed hendecagon: (1) take a radius BC, bisect it at D, make the chord BA equal to the side of the regular 17-gon, and assume DA the side of the 11-gon; and (2) one-fifth the diagonal of the circumscribed square.

Solution by H. J. Woodall, A.R.C.S.



(1) Side of 11-gon $= 2r\sin\frac{1}{11}\pi$,

side of 17-gon $= 2r\sin\frac{1}{17}\pi$.

In fig., $AB^2+AC^2 = 2AD^2+2DC^2$;

$\therefore \quad AD^2 = \frac{1}{4}(2AB^2+AC^2)$.

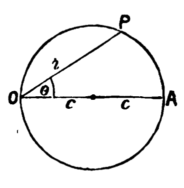
$AD = \frac{1}{2}r\,(1+8\sin^2\frac{1}{17}\pi)^{\frac{1}{2}} = \cdot 5635r$,
which is a trifle too much.

$AD = 2r\sin\frac{1}{11}\pi = \cdot 56346\,.\,r$.

(2) $AD = \frac{1}{5}\times 2\sqrt{2}\,.\,r = r\times\frac{1}{5}(2\cdot 8284) = \cdot 5657r$, ·4 per cent. too much.
[Mr. Pressland finds the angle to be in (1) 32° 44′ 29″, in (2) 32° 41′ 35″, instead of 32° 43′ 38″, which is the true angle.]

11685. (Professor Neuberg.)—On divise l'aire d'un cercle en n parties égales au moyen de cordes issues d'un point fixe de la circonférence. Trouver la moyenne arithmétique des longueurs de ces cordes lorsque n croît indéfiniment.

Solution by C. Morgan, M.A.; H. W. Curjel, B.A.; *and others.*



Mean length of chord

$$= 1/n\,.\,(r_1+r_2+\ldots)$$

$$= \frac{1/n\,.\,(r_1+r_2+\ldots)\,\frac{1}{2}r^2d\theta}{\frac{1}{2}r^2d\theta}$$

$$= \frac{\frac{1}{2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} r^3d\theta}{\pi c^2} = \frac{4c^3\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos^3\theta\,d\theta}{\pi c^2} = \frac{16c}{3\pi}.$$

11725. (Professor HUMBERT.)—Si l'on pose $\tan \phi/n = t$, on a

$$\tan \phi = \frac{C_n' t - C_n^3 t^3 + C_n^5 t^5 - \dots}{1 - C_n^2 t^2 + C_n^4 t^4 - \dots} = \frac{P}{Q}.$$

Montrer que l'on a

$$(1)\ P^2 + Q^2 = (1+t^2)^n, \quad \text{et} \quad (2)\ Q^2 - P^2 = 1 - C_n^2 t^2 + C_{2n}^4 t^4 - \dots;$$

et (3) déduire de là des relations entre C_n^p et C_{2n}^q.

Solution by H. J. WOODALL, A.R.C.S.

From the given values of

$$P\,(= C_n^1 t - C_n^3 t^3 + \dots), \quad \text{and} \quad Q\,(= 1 - C_n^2 t^2 + C_n^4 t^4 - \dots),$$

we get $\qquad Q + iP = (1+it)^n, \quad Q - iP = (1-it)^n;$

therefore $Q = \frac{1}{2}\left\{(1+it)^n + (1-it)^n\right\}, \quad iP = \frac{1}{2}\left\{(1+it)^n - (1-it)^n\right\};$

therefore (1) $Q^2 + P^2 = (Q+iP)(Q-iP) = (1+it)^n(1-it)^n = (1+t^2)^n,$

$$(2)\ Q^2 - P^2 = \tfrac{1}{2}(1+it)^{2n} + \tfrac{1}{2}(1-it)^{2n} = 1 - C_{2n}^2 t^2 + C_{2n}^4 t^4 - \dots.$$

(3) These relations (between C_n^p and C_{2n}^q) may be best obtained thus:—

$$(1+t)^{2n} = (1 + C_n' t + C_n^2 t^2 + \dots)^2 = (1 + C_{2n}' t + C_{2n}^2 t^2 + \dots),$$

if $q = 2r$ equate coefficients of t^{2r} on both sides; therefore

$$C_{2n}^{2r} = (C_n^r)^2 + 2\,(C_n^{r-1} C_n^{r+1} + C_n^{r-2} C_n^{r+2} + \dots),$$

so, also, $\qquad C_{2n}^{2r+1} = 2\,(C_n^r C_n^{r+1} + C_n^{r-1} C_n^{r+2} + \dots).$

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440. (J. W. ELLIOTT, M.A.)—Given two concentric conics; then, if lines be drawn from each point in one of them to touch the other, the chords of contact will touch another curve of the second order.

Solution by J. C. ST. CLAIR.

The theorem that the polar reciprocal of one conic with respect to another is a third conic is true generally (SALMON, Art. 303; PONCELET, I. 231). It may be shown, however, that if the conics are concentric the third conic is concentric with them. For, if ABCD are the intersections of two concentric conics X, Y, the tangents at these points to Y are the polars of the points, considered as belonging to X, and are therefore tangents to the third conic Z. But these tangents obviously form a parallelogram having the same centre as Y; and therefore Z, which is inscribed in it, must also have the same centre.

11679. (Elizabeth Blackwood.) — On a straight line of length $a+b+c$ are measured at random two segments of lengths $a+b$, $b+c$, respectively. Prove that, if a be $>$ b, the mean value of the common segment is $b+c-b^2/3a$.

Solution by H. W. Curjel, B.A.

If $a>c$ the mean value of common part of segments of lengths $a+b$, $b+c$

$$=\frac{\displaystyle\int_0^c\left[\int_0^x(b+c-x+y)\,dy+\int_x^{x+a-c}(b+c)\,dy+\int_{x+a-c}^{a}(x+a+b-y)\,dy\right]dx}{\displaystyle\int_0^c dx\int_0^a dy}$$

$$=\frac{1}{ac}\int_0^c\left[\tfrac{1}{2}(b+c)^2-\tfrac{1}{2}(b+c-x)^2+(a-c)(b+c)-\tfrac{1}{2}(x+b)^2+\tfrac{1}{2}(b+c)^2\right]dx$$

$$=\frac{1}{ac}\int_0^c\left[(b+c)(a+b)-\tfrac{1}{2}(b+c-x)^2-\tfrac{1}{2}(x+b)^2\right]dx$$

$$=\frac{1}{ac}\left[(b+c)(a+b)\,c-\frac{3b^2c+3bc^2+c^3}{3}\right]=b+c-\frac{c^2}{3a}.$$

10405. (R. W. D. Christie.)—Prove that

$$(13p-4^n.r).10^{6m+n}+r\equiv 13.\mathrm{R},$$

where m, n, p, r may be any integers whatever.

Solution by Professors Ignacio Beyens, Morel, *and others*.

On a $(13p-4^n.r)\,10^{6m+n}+r=13.p.10^{6m+n}-4^n.10^{6m+n}.r+r$

$$=(13)-r\,(40^n.10^{6m}-1);$$

mais $40^n=(m.13+1)^n=m.13+1$ et $10^{6m}=(m.13+1)^m=m.13+1$;

donc $40^n.10^{6m}=m.13+1$, et par suite

$$(13p-4^n.r).10^{6m+n}+r=m.13+m.13+1-1=m.13.$$

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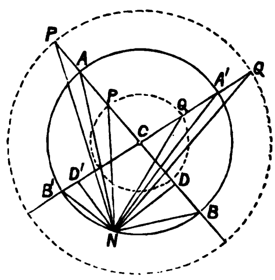
553. (J. H. Swale.)—If, through P, Q, any two points equally distant from the centre (C) of a given circle, there be drawn the diameters AB, A′B′, and from P, Q; A, B: A′, B′ the right lines PN, QN; AN, BN; A′N, B′N to any point N whatever in the periphery, then will

$$\mathrm{PN}^2-\mathrm{PA}^2:\mathrm{Q'N}^2-\mathrm{QA'}^2=\mathrm{AN}^2:\mathrm{A'N}^2,$$

and $$\mathrm{BP}^2-\mathrm{PN}^2:\mathrm{B'Q}^2-\mathrm{QN}^2=\mathrm{BN}^2:\mathrm{B'N}^2.$$

Solution by Professors SHIELDS, ZERR, *and others.*



$$PN^2 = ND^2 + (AD \pm AP)^2,$$
$$PN^2 - PA^2 = ND^2 + AD^2 \pm 2AP \,.\, AD$$
$$= AD\,(AB \pm 2AP) \quad \ldots\ldots(1),$$
$$QN^2 = ND'^2 + (A'D' \pm A'Q)^2,$$
$$QN^2 - QA'^2 = ND'^2 + A'D'^2 \pm 2A'Q \,.\, A'D'$$
$$= A'D'\,(AB \pm 2A'Q) \quad \ldots(2).$$

Since $AP = A'Q$, (1) ÷ (2) gives

$$\frac{PN^2 - PA^2}{QN^2 - QA'^2} = \frac{AD}{A'D'} = \frac{AB \,.\, AD}{A'B' \,.\, A'D'} = \frac{AN^2}{A'N^2};$$

therefore $\quad PN^2 - PA^2 = QN^2 - QA'^2 = AN^2 : A'N^2.$

Similarly, for the second proportion, the plus and minus signs are respectively used when P, Q are *without* and *within* the circle.

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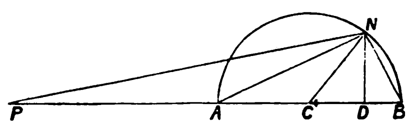
549. (J. H. SWALE.)—If, in the diameter (AB), or in the diameter produced, of a given circle, and on the same side of the centre (C) with A, there be taken a given point (P); and right lines AN, BN, PN be drawn from A, B, P to any point N whatever in the periphery, then will $AN^2 : PN^2 - PA^2 = BN^2 : BP^2 - PN^2$ be a given constant ratio.

Solution by W. J. GREENSTREET, M.A.; Professor ZERR; *and others.*

Let ND be perpendicular to AB; then, since

$$BP + PA$$
$$= BP + PD - DA,$$



$$\frac{AB}{BD} = \frac{AB\,(PB + PA)}{BD\,(BP + PD) - AD \,.\, DB} = \frac{BP^2 - PA^2}{BP^2 - PD^2 - DN^2} = \frac{BP^2 - PA^2}{BP^2 - PN^2},$$

$$\frac{AB}{BD} = \frac{AB^2}{AB \,.\, BD} = \frac{AN^2 + NB^2}{NB^2}; \qquad \therefore \quad \frac{AN^2 + NB^2}{NB^2} = \frac{BP^2 - A^2}{BP^2 - PN^2};$$

$$\therefore \quad \frac{AN^2}{BN^2} = \frac{PN^2 - PA^2}{BP^2 - PN^2}; \quad \text{whence} \quad \frac{AN^2}{PN^2 - PA^2} = \frac{BN^2}{BP^2 - PN^2},$$

and each of these ratios $= AB^2 : BP^2 - AP^2 =$ constant (A, P, B being fixed points).

11942. (D. BIDDLE.)—The series consisting of the reciprocals of figurate numbers of the third order (1, 3, 6, 10, &c.) is summed, from the n^{th} term onwards to infinity, by $2/n$; or if the number of the first term be not known, but this and the next term be given, then

$$\Sigma^{\infty}\ \frac{1}{a}+\frac{1}{a+b}+\text{\&c.}=\frac{b}{a}.$$

But n can be found from a alone, although to have to do so renders the summation a more lengthy process.

Solution by H. W. CURJEL, B.A.; Professor MOREL; *and others.*

$$u_n=\frac{2}{n\,(n+1)}=\frac{2}{n}-\frac{2}{n+1};\quad \text{therefore } \Sigma_n^{\infty}\,\frac{2}{n\,(n+1)}=\frac{2}{n}.$$

If $$a=\frac{n\,(n+1)}{2},\quad \text{and}\quad a+b=\frac{(n+1)(n+2)}{2},\quad \frac{b}{a}=\frac{2}{n};$$

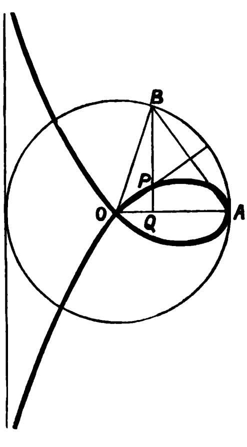
therefore $$\Sigma^{\infty}\ \frac{1}{a}+\frac{1}{a+b}+\ldots=\frac{b}{a}.$$

[Moreover, since $a=\frac{1}{2}n\,(n+1)$, the last clause of the question is explained; for we have $n^2+n=2a$, whence $n=\frac{1}{2}\left\{(8a+1)^{\frac{1}{2}}-1\right\}$, and

$$\Sigma^{\infty}\ 1/a+\ldots-4/\left\{(8a+1)^{\frac{1}{2}}-1\right\}.]$$

11947. (W. J. GREENSTREET, M.A.)—If AB be a chord of a circle centre O, find the locus of the orthocentre of the triangle OAB as AB turns round O.

Solution by W. J. DOBBS, B.A.; R. KNOWLES, B.A.; *and others.*



Let BQ be perpendicular on OA, and OP perpendicular on BA. Then, if

$$\angle \text{POA}=\theta$$

and $$\text{OP}=r,$$

and radius of circle $=a$,

we see that $$\angle \text{AOB}=2\theta;$$

$$\therefore\ \text{OP}\cos\theta=\text{OQ}=\text{OB}\cos 2\theta;$$

$$\therefore\ r=\frac{a\cos 2\theta}{\cos\theta},$$

which is the polar equation of the locus required.

11968. (Professor ROGEL.)—Find the values of the infinite products,

$$(1)\quad \prod_{p=2} p^{(1/p)\sin 2p\mu\pi}, \qquad \text{especially}$$

$$\frac{5^{\frac{1}{5}}\,.\,9^{\frac{1}{9}}\,.\,13^{\frac{1}{13}}\ldots(4n+1)^{1/(4n+1)}}{3^{\frac{1}{3}}\,.\,7^{\frac{1}{7}}\,.\,11^{\frac{1}{11}}\,.\,15^{\frac{1}{15}}\ldots(4n+3)^{1/(4n+3)}}, \qquad \frac{4^{\frac{1}{4}}\,.\,7^{\frac{1}{7}}\,.\,10^{\frac{1}{10}}\ldots(3n+1)^{1/(3n+1)}}{2^{\frac{1}{2}}\,.\,5^{\frac{1}{5}}\,.\,8^{\frac{1}{8}}\,.\,11^{\frac{1}{11}}\ldots(3n+2)^{1/(3n+2)}};$$

$$(2)\quad \prod_{p=3} p^{1/[(-1)^{\frac{1}{2}(p-1)}p-1]}; \qquad (3)\quad \prod_{p\equiv 1 \bmod 4} p^{p/(p^2-1)}\,.\prod_{p\equiv 3 \bmod 4} p^{-1/(p^2-1)}.$$

Solution by the PROPOSER.

Subtracting $\log 2 \sum\limits_{p=1}^{\infty} 1/p \sin 2p\mu\pi = \frac{1}{2}\pi(1-2\mu)\log 2,\ 0<\mu<1$

from
$$\sum_{p=1} \frac{\log 2p}{p}\sin 2p\mu\pi$$
$$= \pi\left\{\log\Gamma(\mu)-(1-\mu)\log\pi - c\left(\tfrac{1}{2}-\mu\right)+\tfrac{1}{2}\log\sin\mu\pi\right\}, \quad c = 0{\cdot}557\ldots,$$
it becomes
$$\sum_{p=1}^{\infty}\frac{\log p}{p}\sin 2p\mu\pi = \pi\left\{\log\Gamma(\mu)-(1-\mu)\log\pi - c\left(\tfrac{1}{2}-\mu\right)+\tfrac{1}{2}\log\sin\mu\pi \right.$$
$$\left. -\left(\tfrac{1}{2}-\mu\right)\log 2\right\}, \quad 0<\mu<1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Both sides are convertible into the logarithm of a product; hence

$$\prod_{p=2}^{\infty} p^{1/p\sin 2p\mu\pi} = \left(\frac{(\sin\mu\pi)^{\frac{1}{2}}\,.\,\Gamma(\mu)}{2^{\frac{1}{2}-\mu}\;{}^{1-\mu}\,e^{(\frac{1}{2}-\mu)c}}\right)^{\pi} \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The evaluation of this *infinite product* depends therefore only on that of the function $\Gamma(\mu)$, and is known, but for the specific values

$$\mu=\tfrac{1}{2},\ \Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}; \quad \mu=\tfrac{1}{4},\ \Gamma\left(\tfrac{1}{4}\right)=2\pi^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}},\ k=\sqrt{\tfrac{1}{2}};$$
$$\mu=\tfrac{1}{3},\ \Gamma\left(\tfrac{1}{3}\right)=2^{\frac{7}{9}}\,.\,3^{-\frac{1}{12}}\,.\,\pi^{\frac{1}{3}}\,.\,\mathrm{K}_1^{\frac{1}{3}},\ k_1=\sin\tfrac{1}{12}\pi$$

(BIGLER, *Crelle's Jour.*, CII., p. 237),

where K, K_1 denotes the complete elliptical integral with respect to the modulus k resp. k_1.

Since $\quad \sin 2p(1-\mu\pi) = -\sin 2\mu p\pi, \quad \sin(1-\mu)\pi = \sin\mu\pi,$

it is sufficient to consider only the assumptions $\mu \leqq \frac{1}{2}$.

(*a*) $\mu=\frac{1}{4}$. We have

$$\frac{5^{\frac{1}{5}}\,.\,9^{\frac{1}{9}}\,.\,13^{\frac{1}{13}}\ldots(4n+1)^{1/(4n+1)}}{3^{\frac{1}{3}}\,.\,7^{\frac{1}{7}}\,.\,11^{\frac{1}{11}}\,.\,15^{\frac{1}{15}}\ldots(4n+3)^{1/(4n+3)}}\left(2\,\frac{\mathrm{K}}{\pi e^{\frac{1}{2}c}}\right)^{\frac{1}{2}\pi} \ldots\ldots\ldots\ldots(3).$$

(*b*) $\mu=\frac{1}{3}$. We get, with regard to

$$\sin\tfrac{1}{3}\pi = \sin(6\lambda+8)\tfrac{1}{3}\pi = \sqrt{\tfrac{3}{4}}, \quad \sin(6\lambda+4)\tfrac{1}{3}\pi = -\sqrt{\tfrac{3}{4}},$$

$$\frac{4\ \ .\,7\ \ .\,10^{\ 1}\ldots(3n+1)^{1/(3n+1)}}{2\ \ .\,5^{\frac{1}{5}}\,.\,8^{\frac{1}{8}}\,.\,11\ \ \ldots(3n+\ \ )^{1/(3n+2)}} = \left(2\ \ .\,3^{\frac{1}{4}}\frac{\mathrm{K}_1}{\pi e^{\frac{1}{3}c}}\right)^{2\pi/3\sqrt{3}} \ldots\ldots\ldots(4).$$

Decomposing the roots of the powers in (3) into their prime factor , and then uniting all the terms with the same prim-root p to a sing e

term, the latter will have the exponent

$$(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots)\left(\frac{(-1)^{\frac{1}{2}(p-1)}}{p}+\frac{1}{p^2}+\frac{(-1)^{\frac{1}{2}(p-1)}}{p^3}+\ldots\right)$$
$$=\tfrac{1}{4}\pi\frac{1}{(-1)^{\frac{1}{2}(p-1)}p-1}.$$

In this product, so transformed, p assumes all the prime numbers except 1 and 2. Involving finally both sides in the $(4/\pi)^{\text{th}}$ power, we obtain

$$\prod_{p=3}^{\infty}\frac{1}{p^{(-1)^{\frac{1}{2}(p-1)}p-1}}=\frac{4K^2}{\pi^2 e^c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The product in (4) may be transformed in the same manner. The arbitrary prime number p obtains the exponent

$$(1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{7}\ldots)\left(\frac{\epsilon}{p}+\frac{1}{p^2}+\frac{\epsilon}{p^3}+\ldots\right),$$

where in the first factor all the fractions whose denominators $\equiv \epsilon$, mod 3, have the negative sign, and where $\epsilon=+1$ if $p\equiv 1$, mod 3, and $=-1$ if $p\equiv 2$, mod 3; in general $\epsilon=\dfrac{2}{\sqrt{3}}\sin\dfrac{2p\pi}{3}$. The first factor proceeds also from $\tfrac{1}{2}(\pi-u)=\sum_{r=1}^{\infty}\dfrac{1}{r}\sin ru$ by setting $u=\dfrac{2\pi}{3}$, then

$$\tfrac{1}{6}\pi=\sqrt{\tfrac{3}{4}}\,(1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{7}\ldots)\,; \quad \text{hence } 1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{7}\ldots=\frac{\pi}{3\sqrt{3}}.$$

Therefore the required exponent will be $\dfrac{\pi}{\sqrt{3}}\,\dfrac{1}{\epsilon p-1}$. The variable p assumes in this infinite product all the prime numbers except 1 and 3. Having involved both sides in the $\left(\dfrac{\pi}{3\sqrt{3}}\right)^{\text{th}}$ power, it becomes

$$\prod_{p=2,5,7\ldots}^{\infty}p^{1/(\epsilon p-1)}=\frac{2^{\frac{2}{3}}\,.\,3\,.\,K_1^2}{\pi^2 e^c}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Dividing the equation (5) by the latter, we eliminate by this means all the terms whose roots are at the same time either $\equiv 1$, mod 4, or $\equiv -1$, mod 4; $\equiv 1$, mod 3, or $\equiv -1$, mod 3. The exponent of the other prime numbers will be

$$\{p-1+(-1)^{\frac{1}{2}(p-1)}(p+1)\}/(p^2-1)=+\frac{2p}{p^2-1},\ \text{if } p\equiv 1,\ \text{mod } 4,$$
$$=-\frac{2}{p^2-1},\ \text{if } p\equiv -1,\ \text{mod } 4.$$

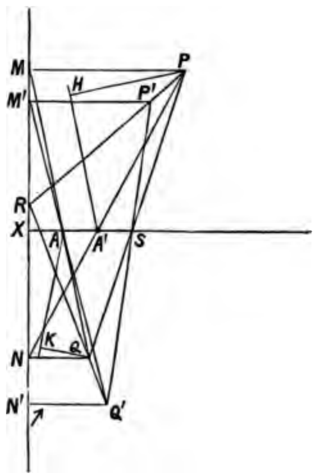
Dividing both sides by $2^{\frac{1}{3}}\,.\,3^{-\frac{1}{2}}$, and extracting the square root, we get,

$$\text{finally,}\quad \prod_{p\equiv\text{mod }4}^{\infty}p^{p/(p^2-1)}\,.\prod_{p\equiv 3\text{ mod }4}^{\infty}p^{-1/(p^2-1)}=2^{\frac{1}{2}}\,.\,3^{-\frac{3}{4}}\,.\,\frac{K}{K_1}\quad\ldots\ldots\ldots(7),$$

where in the first product p denotes prime numbers $\equiv 1$, mod 4, and in the second those $\equiv -1$, mod 4, except the numbers 1, 2, 3, and, in both cases, those by which $p\equiv\pm 1$, mod 12.

7539. (Professor WOLSTENHOLME, M.A., Sc.D.) — (Generalization of Question 6809; solved in *Reprint*, Vol. 36, p. 100.)—Four points S, A′, A, X are taken on a straight line, so that SA′ = AX; the point S will be called the focus, and the straight line through X at right angles to SX the directrix. Any point P being taken in the plane, another point Q is determined as follows: PM is let fall perpendicular on the directrix and SP, AM intersect in Q; prove that (1) the loci of P, Q will be curves of the same order and class; (2) the tangents at P, Q will always intersect on the directrix; (3) if QN be let fall perpendicular on the directrix, N, P, A′ will be collinear; (4) if the locus of P be a conic having the given focus and directrix, so also will the locus of Q; (5) if the locus of P be a parabola having S for focus and A′ for vertex, that of Q will be a parabola with S focus and A vertex; (6) if the tangents at P, Q include a given angle, their loci will be both parabolas with focus S, A′A points on the tangents at their vertices and directrices making the given angle with SX in opposite directions; or, will be corresponding tangents to these two parabolas.

Solution by H. W. CURJEL, B.A.



Let PN cut SX in A″; then, from similar triangles,

$$XA : QN = MX : MN = PS : PQ = A''S : QN;$$

$\therefore\ XA = A''S$;

$\therefore\ A''$ coincides with A′;

i.e., N, P, A′ are collinear ... (3).

Let a point P′ determine M′, Q′, N′ as M, Q, N were determined by P, and let PP′, QQ′ cut the directrix in p and q.

Now $PM : XA' = MN : XN$;

$P'M' : XA' = M'N' : XN'$.

$QN : XA = MN : XM$,

$Q'N' : XA = M'N' : XM'$.

And
$$pM = \frac{MM' . PM}{PM - P'M'} = \frac{MM' . MN . XN'}{MN . XN' - M'N' . XN},$$

$$qN = \frac{NN' . QN}{Q'N' - QN} = \frac{NN' . MN . XM'}{M'N' . MX - MN . XM'};$$

$$\therefore\ \frac{pM + qN}{MN} = \frac{(XM - XM')\,XN'}{XN'(XM + XN) - XN\,(XM' + XN')} + \frac{XM'\,(XN' - XN)}{XM\,(XM' + XN') - XM'\,(XM + XN)} = 1.$$

$\therefore\ p$ and q coincide; *i.e.*, PP′, QQ′ meet on the directrix.

Hence we see that, if Q″ corresponds to P″, which is collinear with

PP′, Q″ is collinear with QQ′; therefore in the figures (P) and (Q) straight lines correspond uniquely to straight lines, and points to points; therefore the loci of P and Q will be curves of the same order and the same class .... (1).

Also corresponding straight lines cut on the directrix; therefore the tangents at P and Q cut on the directrix .... (2).

If PP′ lie on a conic with S and XM as focus and directrix,

$$\frac{PS}{P'S} = \frac{PM}{P'M'} = \frac{MN}{M'N'} \cdot \frac{XN'}{XN}.$$

But
$$\frac{PS}{QS} = \frac{XM}{XN}, \quad \frac{P'S}{Q'S} = \frac{XM'}{XN'};$$

$$\therefore \quad \frac{QS}{Q'S} = \frac{PS}{P'S} \cdot \frac{XM' \cdot XN}{XM \cdot XN'} = \frac{MN}{M'N'} \cdot \frac{XM'}{XM} = \frac{QN}{Q'N'};$$

∴ locus of Q is a conic, with S and XM as focus and directrix ...... (4).

To a point P on the line at infinity there evidently corresponds a point Q on the perpendicular to AX through A, also to a point P on the perpendicular through A′ to AX corresponds a point Q at infinity, and to a point P on any straight line through S corresponds a point Q on the same straight line; also S and all points on XM are self-corresponding.

Hence, if the locus of P is a parabola, with A′ as vertex and S as focus, the locus of Q is a parabola with A as vertex and axis XA. Also, if S is the focus of the locus of P, the circulars of S touch the locus of P, and locus of Q; therefore S is the focus of the locus of Q .... (5).

If the tangents at P and Q cut at a constant angle, then the circle through P and Q and the point of intersection R of their tangents must touch the directrix in R;

$$\therefore \quad \angle PRM = \angle PQR, \text{ and } \angle QRN = \angle QPR;$$

hence, if $\angle SA'H$ be made equal to $\angle QRP$, and $\angle SAK = \angle PRQ$ (considering the sign of the angle in each case), it is evident that, if PH and QK are drawn perpendicular to A′H, AK, PR and QR bisect the angles SPH and SQK. Hence the locus of P is a parabola, with focus S and directrix parallel to A′H, and the locus of Q a parabola, with focus S and directrix parallel to AK. Also, since the line at infinity touches the locus of Q, the line through A′ perpendicular to SA′ is a tangent to the locus of P; therefore A′ is on the tangent at the vertex; therefore A′H is the tangent at the vertex to the locus of P. Similarly AK is the tangent at the vertex to the locus of Q .... (6).

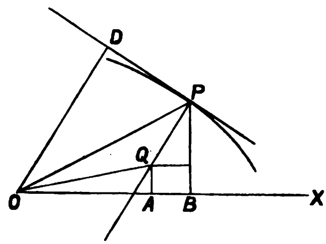
If R is a fixed point, the loci of P and Q are corresponding tangents to the parabolas.

[The figure (Q) is evidently the projection of the figure (P); S, XM, and the line through A′ perpendicular to AX, being centre of projection, axis of projection, and vanishing line.]

10406. (Editor.)—If at each point P of the curve $r = 8a\cos^3 \frac{1}{3}\theta$ we take a length $PQ = 3a$ along the normal at P towards the centre of

curvature, the locus of Q will be a two-cusped epicycloid generated by a circle of radius a rolling on a fixed circle of radius $2a$, the centre of this fixed circle being at a distance a from the origin. [The evolute of the curve $r = 8a\cos^3\frac{1}{3}\theta$ is a similar epicycloid of half the linear dimensions.]

Solution by Professors ZERR, MADHAVARAO, *and others.*



Let P be any point of the given curve, O its origin, DP tangent at point P, OP perpendicular from origin on tangent, PQ normal at point P, PQ = $3a$.

From the equation to the curve,

$$dr/rd\theta = -\tan\tfrac{1}{3}\theta;$$

$$\therefore \quad \angle POD = \angle OPQ = \tfrac{1}{3}\theta.$$

$$(OQ)^2 = r^2 + 9a^2 - 6ar\cos\tfrac{1}{3}\theta$$

$$= 64a^2\cos^6\tfrac{1}{3}\theta + 9a^2 - 48a^2\cos^4\tfrac{1}{3}\theta \quad \text{.........} \quad (1).$$

Let $\qquad OA = x,\ AQ = y,\ OB = x_1,\ PB = y_1.$

Then $\qquad (x_1 - x)^2 + (y - y)^2 = 9a^2 \quad \text{........................} \quad (2).$

But $\qquad x_1 = r\cos\theta, \quad y_1 = r\sin\theta.$

Hence, from (2),

$$x^2 + y^2 + 64a^2\cos^6\tfrac{1}{3}\theta - 16a(x\cos\theta + y\sin\theta)\cos^3\tfrac{1}{3}\theta - 9a^2 \text{......} \quad (3).$$

From (1) $\qquad x^2 + y^2 = 64a^2\cos^6\tfrac{1}{3}\theta + 9a^2 - 48a^2\cos^4\tfrac{1}{3}\theta \quad \text{...............} \quad (4).$

Eliminating θ between (3) and (4) by making $\cos\frac{1}{3}\theta = u$,

we get $\qquad (x^2 - 2ax + y^2 - 3a^2)^3 = 108a^4(x-a)^2;$

changing the origin to $(a, 0)$,

$$(x^2 + y^2 - 4a^2)^3 = 108a^4x^2,$$

the equation of the bicusped epicycloid generated by a circle of radius a, rolling on a fixed circle of radius $2a$, having its cusps at the extremities of the vertical diameter of the fixed circle. For the evolute

$$dr/d\theta = -8a\cos^2\tfrac{1}{3}\theta\sin\tfrac{1}{3}\theta = r\tan(\theta - \phi),$$

where ϕ has the usual meaning in the intrinsic equation;

$$\therefore \quad \tfrac{1}{3}\theta = \tfrac{1}{4}\phi.$$

For the intrinsic equation to the evolute of this curve,

we have $\qquad s = C - \rho = C - ds/d\phi;$

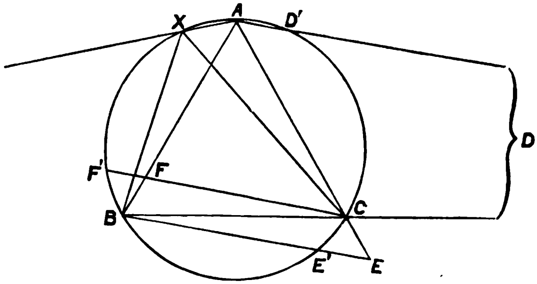
$$\therefore \quad s = C - 6a\cos^2\tfrac{1}{4}\phi = 6a(1 - \cos^2\tfrac{1}{4}\phi) = 6a\sin^2\tfrac{1}{4}\phi.$$

$s = 6a\sin^2\frac{1}{4}\phi$ is the intrinsic equation to a bicusped epicycloid generated by a circle of radius $\frac{1}{2}a$, rolling on a fixed circle of radius a, having its cusps at the extremities of the horizontal diameter of the fixed circle.

11281. (A. J. PRESSLAND, M.A.)—If three parallel lines be drawn through the vertices of a triangle, prove that their isotoms intersect on the minimum ellipse.

Solution by H. W. CURJEL, B.A.

The triangle may be projected orthogonally into an equilateral triangle, and the minimum ellipse becomes the circumcircle.



Let AD, BE, CF be three parallel straight lines through the vertices A, B, C of the equilateral triangle ABC, cutting the opposite sides in D, E, F, and the circumcircle in D'E'F'. Let F fall between A and B.

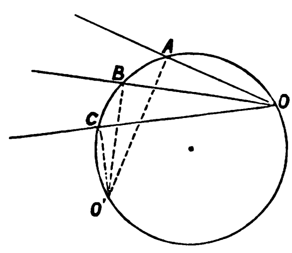
Angles E'CB, ACD' evidently = angle BCA.

Therefore the isotoms of BE, AD cut on the circumcircle at a point X, such that arc AX = arc D'A.

Hence also the isotom of CF cuts the circumcircle in X.

11994. (Professor MORLEY.)—In a plane table are three straight collinear grooves which meet at a point O. A rigid plane lamina has three little projecting feet, A, B, C, which are placed in the grooves. Prove that the lamina can move on the table if the circumcircle ABC passes through O.

Solution by H. W. PYDDOKE, M.A.



Draw perpendiculars at A, B meeting at O'. Then O' must be the centre of motion, when motion is possible; and if C is to move, OCO' must also be a right angle. Thus the condition that motion should be possible is that the perpendiculars at A, B, C should meet in a point.

But if we are given that O is on the circle ABC, then find O', the other end of the diameter through O.

Then OAO', OBO', and OCO' are necessarily right angles, being the angles in a semicircle. Therefore O' is a possible centre of motion, and if force be applied the lamina will *begin* to move. Also, if the circumcircle be supposed to move along with the

triangle, infinitesimal motion round O′ will not move the circle off O, as the tangent at O is the direction of motion, and therefore, after the first infinitesimal motion, further motion will be possible, round a new centre of motion.

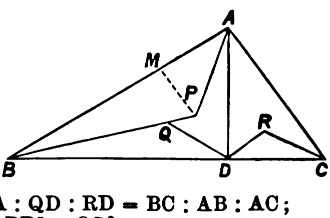
11972. (Professor Bernès.)—De part et d'autre du point A, commun à deux circonférences O et O′, on porte sur le rayon AO, de l'une, deux longueurs égales AL, AM; et, des points L et M comme centres, avec LA, MA comme rayons, on trace deux circonférences qui coupent la circonférence O′ en B′ et C′. Si B et C sont les points où AB′, AC′ rencontrent la circonférence O, et que AD soit une corde de O tangente à O′, et AD′ une corde de O′ tangente à O, démontrer que chacun des quadrilatères ABCD, A′B′C′D′ est harmonique.

Solution by Profs. Droz Farny, Morel, *and others.*

Les points M, A, L et le point R à l'infini sur LM étant harmoniques, on aura
$$\text{faisceau } O'(MALR) = -1.$$
Comme les cordes AC′, AD, AB′, AD′ sont respectivement perpendiculaires sur les cordes O′M, O′A, O′L et O′R, on a aussi
$$A(C'AB'D') = A(CDBA) = -1,$$
ce qui justifie la proposition énoncée.

12004. (Professor Pelletreau.)—Si, dans un triangle rectangle, on abaisse la hauteur relative à l'hypoténuse et qu'on inscrive des cercles dans le triangle primitif et les deux triangles rectangles résultants, la distance des centres des deux derniers cercles est égale à la distance du centre du premier cercle au sommet de l'angle droit du triangle rectangle donné.

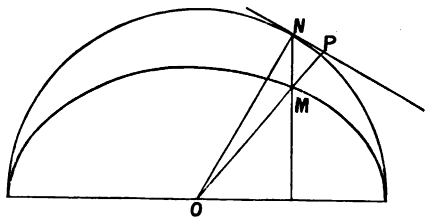
Solution by W. H. Hooker, M.A.; J. Boland; *and others.*



Let A be the right angle; P, Q, R the centres; p, q, r the radii of the in-circles; then, since in $\triangle APM$, $\angle PAM = 45°$,
$$PA = p\sqrt{2};$$
similarly, $DQ = q\sqrt{2}$,
$$DR = r\sqrt{2}.$$
Since triangles are similar,
$$p : q : r = BC : AB : AC, \quad PA : QD : RD = BC : AB : AC;$$
$$\therefore \quad PA^2 = QD^2 + RD^2 = QR^2.$$
Hence the theorem follows.

11866. (Professor MANDART.)—Soient M, N deux points correspondants d'une ellipse et du cercle décrit sur le grand axe comme diamètre. Le rayon OM de l'ellipse rencontrant en P la tangente menée en N au cercle, on demande l'aire de la courbe engendrée par le point P

Solution by H. W. CURJEL, B.A.; Professor BEYENS; *and others.*



The equations to NP and OM are

$$x\cos\phi + y\sin\phi = a,$$

$$\frac{x}{a\cos\phi} = \frac{y}{b\sin\phi};$$

therefore, at P,

$$x = \frac{a^2\cos\phi}{a\cos^2\phi + b\sin^2\phi},$$

$$y = \frac{ab\sin\phi}{a\cos^2\phi + b\sin^2\phi}.$$

In polar coordinates, $r = \dfrac{a\,(a^2\cos^2\phi + b^2\sin^2\phi)^{\frac{1}{2}}}{a\cos^2\phi + b\sin^2\phi}$, $\tan\theta = \dfrac{b}{a}\tan\phi$;

$$\therefore\quad d\theta = \frac{ab\,d\phi}{a^2\cos^2\phi + b^2\sin^2\phi}.$$

The area of locus of P is

$$4\int_0^{\frac{1}{2}\pi} \tfrac{1}{2}r^2 d\theta = 2a^3b\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(a\cos^2\phi + b\sin^2\phi)^2}$$

$$= 2a^3b\int_0^{\infty} \frac{(1+z^2)\,dz}{(a+bz^2)^2}, \text{ where } z = \tan\phi,$$

$$= \frac{a^2(a+b)}{b}\cdot\left(\frac{b}{a}\right)^{\frac{1}{2}}\cdot\frac{\pi}{2} = \pi a^2\cdot\frac{a+b}{2\,(ab)^{\frac{1}{2}}} = \frac{\pi a\,(a+b)}{2}\left(\frac{a}{b}\right)^{\frac{1}{2}}.$$

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876. (MORTIMER COLLINS.)—A person, whose veracity we are unable to determine, asserts that an event has taken place whose simple probability is $= \frac{7}{13}$. Show that the probability of his statement being correct $= \cdot 52567144$.

Solution by Professors ZERR, SARKAR, *and others.*

Let x = probability of the truth of the person's statement; then $\frac{7}{13}x$ and $\frac{6}{13}(1-x)$ = probabilities that the event has or has not taken place;

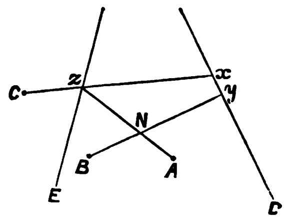
therefore $$p = \int_0^1 \frac{7x/13\,dx}{7x/13 + 6\,(1-x)/13} \div \int_0^1 dx$$

$$= \int_0^1 \frac{7x\,dx}{7x + 6\,(1-x)} = \int_0^1 \frac{7x\,dx}{6+x} = 7 - 42\log_e\left(\tfrac{7}{6}\right) = \cdot 52567144.$$

662. (T. T. WILKINSON.)—**A circle O, and three right lines A, D, C are given in position; it is required to circumscribe the given circle by a triangle DEF whose angular points lie on the given lines.**

Solution by W. J. GREENSTREET, M.A.; Professor MOREL; *and others.*

Let A, B, C be any three points, D and E two lines. We can describe a triangle LMN; having the angle N equal to a given angle, its vertices L, M on the lines D, E, and the sides passing through A, B, and C.



Draw through C any line Czx, meeting D and E in z and x. Joining Az, draw By making with Az an angle N. Then x, y give two homographic divisions on D, the double points of which give two solutions.

Hence we can inscribe in a circle a triangle whose sides shall pass through three given points, by taking three false positions and finding the double points of the two homographic systems of points.

Finally, inscribe in the given circle O a triangle whose sides shall pass through the poles of the three given straight lines. Then the tangent triangle at the angular points of the inscribed triangle will be the triangle required.

[*Otherwise*:—If ABC be the triangle formed by the three straight lines, and A′, B′, C′ be the poles of the lines with respect to the circle, inscribe the ΔPQR to the circle, its sides passing through A′, B′, C′ (*vide* McDOWELL's *Exercises*, pp. 105, 106). Draw tangents at P, Q, R. Then AB is the polar of C′: therefore the tangents at P and Q meet on AB at N, say. Similarly we find L and M. Also, as we can generally inscribe two such triangles as PQR, we can generally find two triangles LMN.]

11617. (Professor NILKANTHA SARKAR.)—Two points are taken at random on a given straight line of length a; prove that the probability of their distance exceeding a given length c ($<a$) is $(1-ca^{-1})^2$.

Solution by Professor ZERR; H. W. CURJEL, B.A.; *and others.*

Let AB $= a$, AP $= x$, then the least value of PQ is given when AQ $= x+c$; hence the required probability is

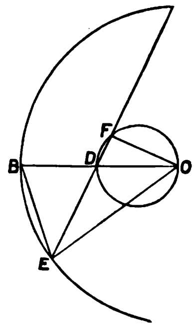
A———P———Q———B

$$p = \int_0^{a-c} x\,dx \bigg/ \int_0^a x\,dx = \left(\frac{a-c}{a}\right)^2.$$

11468. (A. J. PRESSLAND, M.A.)—Examine the following approximation. Take a radius OB of a circle, centre O. Bisect it in D. On OD

as diameter describe a circle. Make the angle DOF $= 22\frac{1}{2}°$, cutting this circle in F. Join FD and produce to cut the larger circle in E. BE is approximately the side of the regular nonagon in the larger circle.

Solution by H. J. WOODALL, A.R.C.S.



In the figure we have the angles

$$\text{OBE} = \text{OEB} = \tfrac{1}{2}\pi - \theta,\quad \text{BOE} = 2\theta,\quad \text{BDE} = \tfrac{3}{8}\pi,$$

$$\text{BED} = \tfrac{1}{8}\pi + \theta,\qquad \text{DEO} = \tfrac{3}{8}\pi - 2\theta,$$

$$\text{BD/DE} = \text{DO/DE};$$

$$\therefore\quad \sin \text{BED}/\sin \text{DBE} = \sin \text{DEO}/\sin \text{DOE};$$

$$\therefore\quad \sin(\tfrac{1}{8}\pi + \theta)/\cos\theta = \sin(\tfrac{3}{8}\pi - 2\theta)/\sin 2\theta;$$

$$\therefore\quad \sin\tfrac{1}{8}\pi + \cos\tfrac{1}{8}\pi\tan\theta = \sin\tfrac{3}{8}\pi\cot 2\theta - \cos\tfrac{3}{8}\pi;$$

$$\therefore\quad 2\sin\tfrac{1}{8}\pi = \cos\tfrac{1}{8}\pi(\cot 2\theta - \tan\theta)$$

$$= \cos\tfrac{1}{8}\pi(1 - 3\tan^2\theta)/2\tan\theta;$$

$$\therefore\quad 3\tan^2\theta + 4\tan\theta\,.\,\tan\tfrac{1}{8}\pi - 1 = 0 \text{ gives}$$

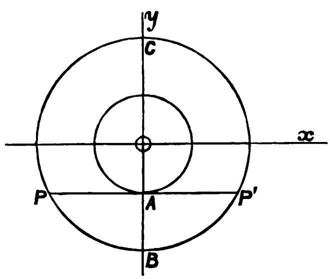
$$\tan\theta = \cdot 363844;$$

whence $\theta = 19°59{\cdot}6'$ nearly, whereas for nonagon θ should be $20°$.

[Mr. PRESSLAND's angle is $19°\,59'\,37{\cdot}75''$.]

12002. (Professor MOREAU.)—D'un point fixe comme centre, on décrit deux cercles dont la différence des rayons est constante. Lieu des extrémités des cordes du grand cercle parallèles à une direction donnée et tangentes au petit cercle.

Solution by V. J. BOUTON, B.Sc.; W. H. HOOKER; *and others.*



Let radius of smaller circle $= r =$ OA,

and that of larger circle $= r + a$.

Then

$$x^2 = \text{PA}^2 = \text{BA}\,.\,\text{AC} = a(2r+a)$$

$$= a(a - 2y);$$

$$\therefore\quad x^2 = a(a-2y).$$

A parabola touching the axis of x at O and vertex downwards.

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135. (KATE SULLIVAN.)—P is a given point in one of the principal axes AB (produced) of a given ellipse; find the position of a secant Pmn,

intersecting the curve at m, n, such that, by drawing the perpendiculars mx, ny on AB, the figure $mxyn$ shall be maximum.

Solution by Professors ZERR, BEYENS, *and others.*

Let (x_1, y_1), (x_2, y_2) be the points m, n; and
$y = \mu(x+c)$, $x^2/a^2 + y^2/b^2 = 1$ be the equations to the secant and ellipse ....(1, 2).

From (1) and (2), x_1, y_1, x_2, y_2 are known; but

$$\tfrac{1}{2}(y_1+y_2)(x_1+x_2) = \left\{2ab^3c\mu\,(b^2-\mu^2c^2+a^2\mu^2)^{\frac{1}{2}}\right\} / (b^2+a^2\mu^2)^2 = max;$$

hence, from the first differential coefficient, we get

$$2a^2(c^2-a^2)\,\mu^4 - b^2(a^2+2c^2)\,\mu^2 + b^4 = 0;$$

therefore $$\mu = \frac{b}{2a}\left\{\frac{a^2+2c^2-(4c^4-4a^2c^2+9a^4)^{\frac{1}{2}}}{c^2-a^2}\right\}$$

gives the position of the secant for maximum.

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995. (EDITOR.)—A box contains 10 sovereigns, 20 shillings, and 40 farthings; what is the probability that in five trials a person will draw half the sovereigns, each coin being replaced after it is drawn? Show that if he receives a guinea for every sovereign that he draws, but pays 5s. for every shilling, and 1s. 6d. for every farthing drawn, the value of his expectation in six trials is $4\frac{2}{7}$ shillings.

Solution by H. W. CURJEL, B.A.

Chance of drawing a sovereign at any trial $= \dfrac{10}{10+20+40} = \dfrac{1}{7}$.

Therefore chance of drawing a sovereign five times in five trials $= (\frac{1}{7})^5$.

Expectation from one drawing

$$= \tfrac{1}{7}\times 21 - \tfrac{2}{7}\times 5 - \tfrac{4}{7}\times\tfrac{3}{2} \text{ shillings} = \tfrac{5}{7} \text{ shillings}.$$

Therefore expectation from six drawings

$$= \tfrac{30}{7} \text{ shillings} = 4\tfrac{2}{7} \text{ shillings}.$$

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764. (MORTIMER COLLINS.)—Form the equations whose roots are the squares of the differences of the roots of $x^3 - 7x + 7 = 0$.

Solution by W. J. GREENSTREET, M.A.; M. BRIERLEY; *and others.*

Let $f(x) = x^3 - 7x + 7 = 0$ have roots α, β, γ. Let $y = (\beta-\gamma)^2$. Then $\phi(x, y) = 0$ is $y = 14 - x^2 + 14/x$ or $x^2 + x(y-14) - 14 = 0$, and, subtracting from $f(x) = 0$, $x = 21/(y-7)$; therefore equation required is

$$21^3 - 147(y-7)^2 + 7(y-7)^3 = 0 \quad \text{or} \quad y^3 - 42y^2 + 441y - 49 = 0.$$

721. (MORTIMER COLLINS.)—From a vessel of wine containing 256 gallons, 64 gallons are drawn off, and the vessel is refilled with water; find how many times this operation must be repeated, so that not more than 1 gallon of pure wine may remain in the vessel.

Solution by W. J. GREENSTREET, M.A.; Professor ZERR; *and others.*

The wine left in after the n^{th} drawing is given by

$$256\left(1-\tfrac{64}{256}\right)^n = 1, \quad \text{or} \quad 3^n = 4^{n-4} = 2^{2n-8};$$

therefore $$n = \frac{8\log 2}{3\log 2 - \log 3} = 17 \text{ to the nearest integer.}$$

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554. (EDITOR.)—If from a point perpendiculars be drawn on the three sides of a given triangle, prove that the area of the triangle formed by joining the feet of the perpendiculars has a constant ratio to the rectangle under the segments of a chord of the circle circumscribed to the given triangle drawn through the point.

Solution by W. J. GREENSTREET, M.A.; Prof. BHATTACHARYA; *and others.*

If Δ' be area of the pedal triangle of P, and O the circumcentre, we have $\Delta' = \frac{1}{2}\Delta - \frac{1}{8}\Sigma AP^2 . \sin 2A = \frac{1}{4}\Delta - \frac{1}{8}(2\Delta + 4OP^2 \sin A \sin B \sin C)$

$$= \tfrac{1}{4}\Delta - \tfrac{1}{8}OP^2 . \frac{abc}{4R} . \frac{1}{2R} = \frac{\Delta}{4R^2}(R^2 - OP^2) = \frac{\Delta}{4R^2}(LP . PK),$$

if OP meet the circumcircle in P and K. Therefore

$$\frac{\Delta'}{LP . PK} = \frac{\Delta}{4R^2} = \text{constant.}$$

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567. (EDITOR.)—A person sitting in a railway carriage throws a ball to the roof in a direction perpendicular to the floor of the carriage. His hand, at the time of the ball leaving it, is 5 feet from the roof of the carriage. The train moves at the rate of 1 mile in 2 minutes, and the ball at the rate of 10 feet in a second. Prove that the angle which the course of the ball makes with the plane of the road is $\tan^{-1}\frac{5}{22}$.

Solution by W. J. GREENSTREET, M.A.; Professor ZERR; *and others.*

The ball has a vertical velocity 10 feet per second, and a horizontal velocity of 44 feet per second; hence the angle required is the one stated.

527. (THOMAS MORLEY.)—A train weighs 100 tons, and is going at the rate of 20 miles per hour when the steam is turned off; find how far the train will go, on a level rail, allowing 8 lbs. per ton friction, the effect of the air being neglected.

Solution by Professor ZERR; W. J. GREENSTREET, M.A.; *and others.*

Let T tons = weight of train, V feet per second its velocity, S the space the train has moved when its velocity is V_0 feet per second, p lbs. per ton = friction. Then work done by friction = pTS, work done by train over space $S = \frac{1}{2} \cdot \frac{2240T}{g}(V^2 - V_0^2)$; therefore

$$pTS = \frac{1}{2} \cdot \frac{2240T}{g}(V^2 - V_0^2), \quad \text{or} \quad S = \frac{1120}{gp}(V^2 - V_0^2);$$

but $\qquad V = 29\frac{1}{3}, \quad V_0 = 0, \quad p = .8, \quad g = 32\frac{1}{8}$

therefore $\qquad S = 37449397$ feet.

11962. (Professor DE LONGCHAMPS.)—Un arc de parabole touche les côtés d'un angle droit yOx aux points A, B. Construire le cercle inscrit au triangle formé par les côtés OA, OB, et l'arc AB.

Solution by R. KNOWLES, B.A.; Professor ZERR; *and others.*

The equation to the circle, referred to Ox, Oy as axes, is

$$x^2 + y^2 - 2r(x+y) + r^2 = 0, \quad \text{or} \quad x + y - (2xy)^{\frac{1}{2}} - r = 0,$$

and that to the parabola $(x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}} = 1$. Substituting, we have

$$x + b(1 - x^{\frac{1}{2}}/a^{\frac{1}{2}})^2 - 2^{\frac{1}{2}}b^{\frac{1}{2}}x^{\frac{1}{2}}(1 - x^{\frac{1}{2}}/a^{\frac{1}{2}}) - r = 0,$$

and, in order that the circle may touch the parabola, the two values of x must be equal; therefore

$$\{a + b + (2ab)^{\frac{1}{2}}\}(b - r)a = ab\{b + (2ab)^{\frac{1}{2}} + a/2\};$$

therefore $\qquad r = ab/2\{a + b + (2ab)^{\frac{1}{2}}\};$

therefore the coordinates of the centre of the circle, each = r, and its radius, are given.

11957. (Professor ZERR.)—A. lends B. \$4000, with the understanding that B. pay him \$5000 in ten equal annual instalments of \$500 each, the first payable at end of first year. Show that the rate of interest is $4\frac{1}{4}$ per cent. nearly.

Solution by R. CHARTRES.

If $R \equiv \$1 +$ its interest for a year, then we have

$$4000 = 500\left\{\frac{1}{R}+\frac{1}{R^2}+\dots\frac{1}{R^{10}}\right\}, \quad \text{or} \quad 8R^{11}-9R^{10}+1=0$$

(introducing the factor $R-1$). $R = \frac{417}{400}$ will make the left-hand member of this a very small negative quantity. Therefore the percentage is extremely near to $4\frac{1}{4}$.

7310. (Rev. T. P. KIRKMAN, M.A., F.R.S.)—From

$$u_{x+1} = x\,(u_x + u_{x-1}),$$

find (1) the integer u_x; and (2) give the simple question in Tactic to which this is the answer.

Solution by H. J. WOODALL, A.R.C.S.

$$u_{x+1} = x\,(u_x+u_{x-1}), \quad \therefore\; u_{x+1}-(x+1)\,u_x = -\left\{u_x - xu_{x-1}\right\} = (-1)^{x+1}\,c;$$

therefore $\qquad u_x = xu_{x-1} + (-1)^x c,$

whence $\qquad u_x = x\,(x-1)\dots(x-n+1)\,u_{x-n}$

$$+(-1)^x c\left\{1-x+x\,(x-1)+\dots+(-1)^{n-1}x\,(x-1)\dots(x-n+2)\right\}.$$

11988. (Professor SYLVESTER.) — If fx, ϕx are any two rational integral functions of x, and the roots of ϕx are all imaginary, prove that the number of real roots in fx, less the number of the same in $\phi x\,.\,f'x - fx\,.\,\phi' x$, cannot be greater than unity. Show also, more generally, that, whatever be the nature of the roots of fx, ϕx, the difference between the number of real roots in the one and the other cannot be greater than the number of the same in $\phi x\,.\,f'x - fx\,.\,\phi' x$, augmented by unity.

Solution by H. W. CURJEL, B.A.

If $f(x) = 0$ has no multiple roots, and if α and β are two real roots of $f(x) = 0$ such that there is no root between them, then $f'(\alpha)$ and $f'(\beta)$ are of opposite signs; hence, if $\phi(x)$ has no root between α and β, $\phi(\alpha)\,f'(\alpha) - f(\alpha)\,\phi'(\alpha)$ and $\phi(\beta)f'(\beta) - f(\beta)\,\phi'(\beta)$ have opposite signs; therefore $\phi(x)f'(x) - \phi'(x)\,f(x) = 0$ has a root between α and β. Hence proposition, when $f(x)$ has no multiple roots. The proposition is easily extended to the case of multiple roots, since, if $f(x)$ has r equal roots each $= \gamma$, $f'(x)$ has $(r-1)$ equal roots each $= \gamma$.

11995. (Professor HUDSON, M.A.)—Show that the number of days elapsed from January 1st, x B.C., to January 1st, y A.D., is

$$(x+y-1)\,365+\mathrm{I}\left(\frac{x+3}{4}\right)+\mathrm{I}\left(\frac{y-1}{4}\right)-\mathrm{I}\left(\frac{y-201}{100}\right)+\mathrm{I}\left(\frac{y-1}{400}\right),$$

where I (m) means the greatest integer in m, and y A.D. is later than the change of style.

Solution by V. J. BOUTON, B.Sc.; Professor BHATTACHARYA; *and others.*

The number of years elapsed is $x+y-1$; hence disregarding leap-years, we have $(x+y-1)\,365$ days. Now, for leap-years B.C., since B.C. 1 is a leap-year, if n is any integer, we must add n days for $x = 4n$, $4n-1$, $4n-2$, $4n-3$ years. That is, we must add $\mathrm{I}\,\frac{1}{4}\,(x+3)$.

For ordinary leap-years A.D., since A.D. 4 is a leap-year, $\mathrm{I}\,\frac{1}{4}\,(y-1)$ days must be added. Now, supposing *all* the "centuries" are ordinary years, we must subtract $\mathrm{I}\,\frac{1}{100}\,(y-1)$ days from the above.

But, each 400th year being a leap-year, we must add $\mathrm{I}\,\frac{1}{400}\,(y-1)$ days.

We must also add 2 days because the correction at the change of style was 2 days short.

Hence, total number of days elapsed is as stated.

[The PROPOSER remarks that this solution is incomplete, as it does not explain why $\mathrm{I}\left(\frac{y-201}{100}\right)$ instead of $\mathrm{I}\left(\frac{y-1}{100}\right)$ occurs. This is because the correction at the change of style was 10 days instead of 12. Also it does not fully explain why it is $x+3$ and $y-1$. See HERSCHELL'S *Astronomy*, pp. 690, 691, of which this is a correction.]

11994. (Professor MORLEY.)—In a plane table are three straight grooves which meet at a point O. A rigid plane lamina has three little projecting feet, ABC, which are placed in the grooves. Prove that the lamina can move on the table if the circumcircle ABC passes through O.

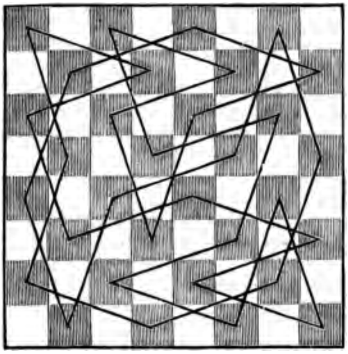
Solution by H. J. WOODALL, A.R.C.S.; Professor BEYENS; *and others.*

Suppose the lamina fixed and the grooves moveable; also that the grooves A′O′B′, B′O″C′ are made separately. Then when A′O′B′ fulfils the conditions, O′A′ passes through A, O′B′ passes through B. But A′O′B′ is an angle of constant magnitude. Therefore the locus of O′ is a certain circle passing through A and B. Similarly the locus of O″ is a circle through B and C. These circles coincide or intersect at two points (B and another) only. In the second case there is only one position of the frame A′B′C′O. If, however, the two circles coincide, the frame may rest in any number of positions, in each of which point O lies on a circle passing through AB (circle O′) which is also on circle O″. Hence the circle is the circumcircle of ABC.

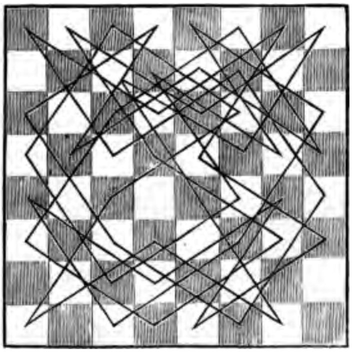
7763. (B. Reynolds, M.A.)—Assuming that a knight's move consists of two castle-moves followed by one bishop-move, construct (1) a reentrant knight's tour, to cover all the black squares of a chess-board; and find (2) how many squares can be covered by a knight whose moves consist of two bishop-moves followed by one castle-move.

Solution by H. W. Curjel, B.A., *and the* Proposer.

1. This may be done by the same method as an ordinary knight's tour, namely, by forming a tour (not complete) generally moving to the square which commands least vacant squares, and then deforming the tour so as to cover the unused squares. Such a tour is given in the annexed figure.

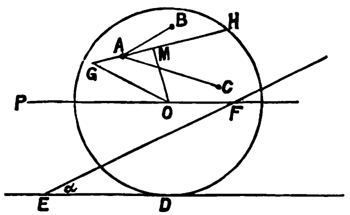


2. A re-entrant tour, covering 46 squares, is shown in the annexed figure.



11246. (Elizabeth Blackwood.)—Three points are taken at random within a sphere on a horizontal plane; find the chance that the plane passing through the three random points will make with the horizontal plane an angle less than a given acute angle a.

Solution by Professor Zerr.



Let GH be the diameter of the section of the sphere made by a plane through the three random points A, B, C; M its centre; O the centre of the sphere; FOP a line parallel to the plane DE such that the line AB is parallel to the plane MOP.

Let $OG = r$, $MA = u$, $AB = v$, $AC = w$, $\angle GOM = \theta$, $\angle BAM = \phi$, $\angle CAM = \psi$, $\angle MOP = \lambda$, and the angle the plane POM makes with some fixed plane through $OP = \rho$.

An element of sphere is, at A, $r\sin\theta\, d\theta\, 2\pi u\, du$; at B, $v^2 dv\, d\phi\, d\lambda$; at C, $\sin(\phi+\psi)\sin\lambda\, w^2 dw\, d\psi\, d\rho$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of u, 0 and $r\sin\theta = u'$, and tripled; of ϕ, $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of ψ, $-\phi$ and $\frac{1}{2}\pi$, and doubled; of λ, $\frac{1}{2}\pi - a$ and $\frac{1}{2}\pi + a$; of ρ, 0 and 2π; of v, 0 and $2u\cos\phi = v'$; of w, 0 and $2u\cos\psi = w'$.

Since $(\frac{4}{3}\pi r^3)^3$ is the whole number of ways the three points can be taken, we have for the required chance

$$p = \frac{6\,.\,3^3}{4^3\pi^3 r^9}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-a}^{\frac{1}{2}\pi+a}\int_0^{2\pi}\int_0^{v'}\int_0^{w'} r\sin\theta\, d\theta\, 2\pi u\, du \sin(\phi+\psi)\, d\phi\, d\psi \times \sin\lambda\, d\lambda\, d\rho\, v^2 dv\, w^2 dw$$

$$= \frac{27}{2\pi^2 r^8}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-a}^{\frac{1}{2}\pi+a}\int_0^{2\pi}\int_0^{v'} \sin\theta\, u^4 \sin(\phi+\psi)\cos^3\psi \sin\lambda\, d\theta\, du\, d\phi\, d\psi \times d\lambda\, d\rho\, v^2 dv$$

$$= \frac{36}{\pi^2 r^8}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-a}^{\frac{1}{2}\pi+a}\int_0^{2\pi} \sin\theta\, u^7 \sin(\phi+\psi)\cos^3\phi\cos^3\psi \sin\lambda\, d\theta\, du \times d\phi\, d\psi\, d\lambda\, d\rho$$

$$= \frac{64}{\pi r^8}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-a}^{\frac{1}{2}\pi+a} \sin\theta\, u^7 \sin(\phi+\psi)\cos^3\phi\cos^3\psi \sin\lambda\, d\theta\, du\, d\phi \times d\psi\, d\lambda$$

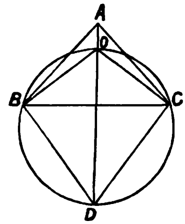
$$= \frac{128\sin a}{\pi r^8}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi} \sin\theta\, u^7 \sin(\phi+\psi)\cos^3\phi\cos^3\psi\, d\theta\, du\, d\phi\, d\psi$$

$$= \frac{16\sin a}{\pi r^8}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \left[3(\tfrac{1}{2}\pi+\phi)\sin\phi + 2\cos\phi + \sin^2\phi\cos\phi\right]\sin\theta\cos^3\phi \times u^7\, d\theta\, du\, d\phi$$

$$= \frac{35\sin a}{2r^8}\int_0^{\frac{1}{2}\pi}\int_0^{u'} \sin\theta\, u^7\, d\theta\, du = \frac{35\sin a}{16}\int_0^{\frac{1}{2}\pi}\sin^9\theta\, d\theta = \tfrac{8}{9}\sin a.$$

11985. (Professor GOPALACHARI.)—A, B, C are three points not in the same straight line. Find a fourth point O, such that the sum of AO, BO, CO is a minimum.

Solution by R. CHARTRES ; W. J. DOBBS, M.A. ; *and others.*



On BC draw an equilateral triangle BCD. Then O, the point required (FERMAT's point), is the intersection of the circumcircle of BCD with AD.

This is the same as Question 11056, Vol. 56.

See also Question 10210, Vol. 54, where an excellent generalized proof is given by Mr. E. M. LANGLEY, M.A.

In that proof, if

(1) $l = m = n$, $OA+OB+OC = AD =$ minimum.

(2) $l : m : n = OA : OB : OC$; then

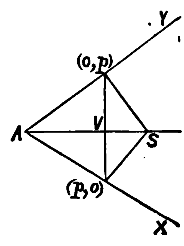
$OA^2+OB^2+OC^2 =$ a minimum = square on the tangent from A.

If $A = 120°$, then O will coincide with A.

If $A > 120°$, A will still be the point.

11763. (Professor SOLLERTINSKY.) — Sur les côtes AX, AY d'un angle donné, on prend deux segments variables AB, AC satisfaisant à la condition $AB+AC = 2p$. Démontrer que la droite BC enveloppe une parabole de paramètre $p \sin \frac{1}{2}A \tan \frac{1}{2}A$; le point de contact de BC est l'isotomique du pied de la bissectrice de l'angle BAC.

Solution by H. W. CURJEL, B.A. ; W. J. GREENSTREET, M.A. ; *and others.*



The equation to BC is

$$x/(p+\lambda)+y/(p-\lambda) = 1,$$

or $\qquad u \equiv p^2-\lambda^2+\lambda(x-y)-p(x+y) = 0$;

$\therefore$ envelope of BC, being result of eliminating λ from $u = 0$ and $du/d\lambda = 0$, is

$$\tfrac{1}{4}(x-y)^2-p(x+y)+p^2 = 0 ;$$

$\therefore$ distance from focus to vertex V

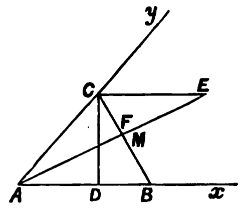
$$= p/\cos\tfrac{1}{2}A-p\cos\tfrac{1}{2}A = p\sin\tfrac{1}{2}A\tan\tfrac{1}{2}A.$$

since A, $(p, 0)$, $(0, p)$ and the focus S are concyclic, and AS bisects the angle YAX.

11859. (Professor NEUBERG.)—Etant donnés un angle xAy et un point M, une droite quelconque menée par M coupe Ax en B, Ay en C ;

on inscrit dans le triangle ABC un carré dont un côté repose sur Ax. Trouver le lieu du sommet du carré situé sur BC.

Solution by R. KNOWLES; Professor BEYENS; *and others.*



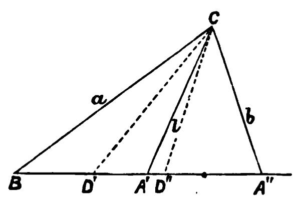
Through C draw CE parallel to AB and = CD. Join AE, meeting BC in F. Then the locus of F is required.

$$\cot \text{FAB} = (\text{AD}+\text{CD})/\text{CD} = 1+\text{AD}/\text{CD} = 1+\cot \text{A}.$$

Therefore the locus required is the line AE, and is independent of the point M.

4178. (R. TUCKER, M.A.)—In the ambiguous case o plane triangles (a, B being fixed and b variable), find the mean area of the triangle contained by the base and the median lines from C.

Solution by H. J. WOODALL, A.R.C.S.; Prof. SARKAR; *and others.*



The length of the base intercepted between the two medians is $b\cos A$; then the height of the triangle is $a\sin B$;

hence area $= \frac{1}{2}ab\sin B\cos A$.

But $b = a\sin B/\sin A$,

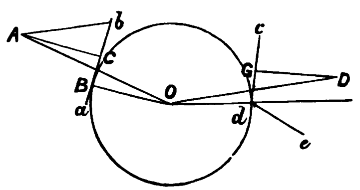
and $db = -\left\{a\sin B\cos A/\sin^2 A\right\} dA$;

$$\therefore \text{ required mean } = \int_{a\sin B}^{a} \tfrac{1}{2}a^2\sin^2 B\cot A\, db \Bigg/ \int_{a\sin B}^{a} db.$$

This becomes
$$= -\frac{a\sin^2 B}{2(1-\sin B)}\int_{\frac{1}{2}\pi}^{B} \frac{\cot A \,.\, a\sin B\cos A\, dA}{\sin^2 A}$$
$$= -\frac{a^2\sin^3 B}{2(1-\sin B)}\int_{\frac{1}{2}\pi}^{B} \frac{\cos^2 A}{\sin^3 A}\, dA$$
$$= \frac{a^2\sin B\cos B + a^2\sin^3 B \,.\, \log\tan\frac{1}{2}B}{4(1-\sin B)}.$$

992. (S. WATSON.)—A circle and square have the same perimeter, and the latter is rolled round the circumference of the former; find the area of the locus of its centre.

Solution by Professor G. B. M. ZERR.



Let a regular polygon of m sides, having same perimeter as circle, roll on the circle.

Let ab be one of the sides of the polygon as it moves from tangency at a to tangency at b; cde a portion of the polygon as it moves from tangency to cd to the position of tangency to de, d remaining at the same point. Let r be the radius of the circle, A, D the centre of the polygon in different positions, C the mid-point of ab, G the mid-point of cd.

$$\angle AOB \text{ or } \angle DOd = \theta, \quad ab = 2\pi r/m, \quad AC = (\pi r/m)\cot \pi/m,$$
$$Ab = (\pi r/m)\operatorname{cosec}\pi/m,$$
$$OA = \rho = (OB + AC)\sec\theta = r/m\,(m + \pi\cot\pi/m)\sec\theta,$$
$$OD = \rho_1 = r\cos\theta + \left\{\pi^2/m^2\operatorname{cosec}^2\pi/m - \sin^2\theta\right\}^{\frac{1}{2}} = \text{a circle}.$$

Each equation represents m areas comprised between the limits $\theta = 0$ and $\theta = \theta' = \tan^{-1}\pi/(m + \pi\cot\pi/m)$, and doubled;

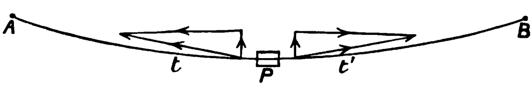
$$\therefore \quad A = m\int_0^{\theta'}\rho^2 d\theta + m\int_0^{\theta'}\rho_1{}^2 d\theta = 2\pi a^2 + \frac{\pi^2 a^2}{m}\left\{\cot\frac{\pi}{m} + \frac{\pi}{m}\operatorname{cosec}^2\frac{\pi}{m}\right\}.$$

Let $\quad m = 4, \quad A = \frac{1}{8}\pi a^2(16 + 2\pi + \pi^2), \quad m = \infty, \quad A = 4\pi a^2.$

9310. (Professor SATIS CHANDRA RAY, M.A.)—Show from statical principles that a string cannot be kept stretched evenly between two points in a horizontal line by any amount of tension unless its mass is infinitesimal.

Solution by D. WATSON, M.A.; Prof. GOPALACHARI; *and others.*

Take a string suspended loosely between two points A, B, which are in a horizontal line. Take a particle of the string small enough to be



regarded as a rigid body. Its equilibrium is maintained by two forces t and t' acting in the directions of the string. Each of these can be resolved into a horizontal and a vertical component, of which the horizontals neutralize each other. The sum of the vertical components supports the particle against gravity. If the string were perfectly horizontal, the vertical components would $= 0$; hence $mg = 0$; or the particle, and hence the string, is without mass.

726. (EDITOR.) — From one of two bags, whereof each contains 4 white and 4 black balls, 4 are taken at random, and transferred to the other bag; then, 8 being drawn from the latter, 6 of them prove *white* and 2 *black*: find the chance that, if another be drawn, it will be *white*.

Solution by D. BIDDLE; K. S. PUTNAM, M.A.; *and others.*

The four balls taken from the first bag may be (1) all white, (2) three white and one black, (3) two white and two black, (4) one white and three black, or (5) all black, and the respective probabilities are $\frac{1}{70}$, $\frac{16}{70}$, $\frac{36}{70}$, $\frac{16}{70}$, $\frac{1}{70}$. But the stated fact, as to the character of the subsequent drawing from the second bag, shows that at least two white balls must have been taken from the first bag, and this excludes the two last probabilities above given. If *only* two white balls were taken, $P = 0$; if three, $P = \frac{1}{4}$; if four, $P = \frac{1}{2}$. Consequently,

$$P = (36 \times 0 + 16 \times \tfrac{1}{4} + 1 \times \tfrac{1}{2})/53 = 9/106 = \cdot 085 \text{ nearly}.$$

[Mr. BIDDLE declines to accept the "stated fact" as indicating more than it clearly does, namely, that at least two white balls were taken from the first bag; but he observes that the respective probabilities of drawing 6 white and 2 black balls from the second bag, according as there are in it, (1) 6 white and 6 black, (2) 7 white and 5 black, (3) 8 white and 4 black, are $\frac{15}{495}$, $\frac{70}{495}$, $\frac{168}{495}$, so that, by the orthodox method, the required probability would be given as

$$P = (36 \times 15 \times 0 + 16 \times 70 \times \tfrac{1}{4} + 1 \times 168 \times \tfrac{1}{2}) / 1828 = \tfrac{91}{457} = \cdot 1991,$$

or nearly $\frac{1}{5}$. He refers the reader to VENN's *Logic of Chance*, in which the introduction of Inverse Probability in this way is shown to be very unsatisfactory; and he regards its use in the present instance as almost tantamount to teaching that the more white balls you take out of the second bag the more you leave in.]

11912. (S. TEBAY, B.A.)—Prove the porismatic identity

$$a^2 + b^2 = \left[\left\{2ars + b\,(r^2 - s^2)\right\}^2 + \left\{2brs - a\,(r^2 - s^2)\right\}^2\right] \div (r^2 + s^2)^2.$$

Solution by J. BOLAND; H. J. WOODALL, A.R.C.S.; *and others.*

We have $$a^2 + b^2 = a^2 + c^2 + b^2 - c^2.$$

Let $a^2 + c^2 = m^2$; therefore $c^2 = m^2 - a^2$, and

$$b^2 - c^2 = a^2 + b^2 - m^2 = \left\{a + r/s\,.\,(b - m)\right\}^2, \text{ suppose};$$

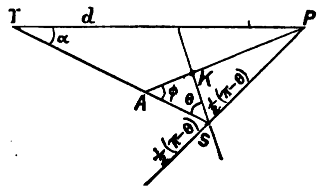
therefore $$m^2 = \left\{2ars + b\,(r^2 - s^2)\right\}^2 \div (r^2 + s^2)^2,$$

and $$b^2 - c^2 = \left\{2brs - a\,(r^2 - s^2)\right\}^2 \div (r^2 + s^2)^2.$$

Therefore, &c. [This follows at once by expansion.]

4496. (T. COTTERILL, M.A.)—A and S are fixed points, P a variable point on a given line d; show that the locus of the intersection of AP with the line of reflexion of AS in SP is the conic of which A is a point, S a focus, and d the corresponding directrix.

Solution by H. J. WOODALL, A.R.C.S.



In the figure, let $AS = a$, $TS = b$, and the angles as marked. Then $SP = b\sin\alpha/\sin\frac{1}{2}(\pi-\theta-2\alpha)$
$= b\sin\alpha/\cos\frac{1}{2}(\theta+2\alpha)$;
hence, taking S as pole and ST as initial line, we find
$r = SK = a\sin\phi/\sin(\theta+\phi)$,
where $a\sin\phi = SP\sin(\phi+\frac{1}{2}\pi+\frac{1}{2}\theta)$.
This becomes $\quad = b\sin\alpha\cos\frac{1}{2}(\theta+2\phi)/\cos\frac{1}{2}(\theta+2\alpha)$;

$$\therefore\quad a\sin\phi\cos\tfrac{1}{2}(\theta+2\alpha) = b\sin\alpha\cos\tfrac{1}{2}(\theta+2\phi),\ \text{ and}$$

$$\cot\phi = \left\{a\cos\tfrac{1}{2}(\theta+2\alpha)+b\sin\alpha\sin\tfrac{1}{2}\theta\right\}/b\sin\alpha\cos\tfrac{1}{2}\theta,$$

$$a/r = \sin\theta\cot\phi+\cos\theta = 1+a\left\{\sin(\theta+\alpha)-\sin\alpha\right\}/b\sin\alpha;\ \text{ whence}$$

$$ab/\left\{(b-a)r\right\} = 1+\left\{a/(b-a)\sin\alpha\right\}\sin(\theta+\alpha),$$

which coincides with $\quad l/r = 1+e\sin(\theta+\alpha)$,

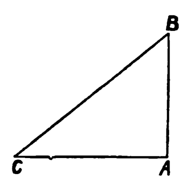
where $\quad l = ab/(b-a)$, and $e = \left\{a/(b-a)\sin\alpha\right\}$,

and axis is perpendicular to d.

Distance of d from $S = b\sin\alpha = l/e$; $\therefore$ d is the directrix.

7162. (Prof. ORCHARD, M.A., B.Sc.)—ABC is a "perfectly rough" inclined plane. When AC is base a sphere rolls down in the same time that a cylinder does when AB is base. Find the angle of the plane.

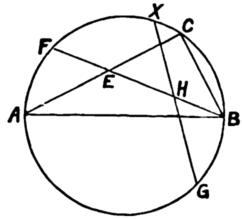
Solution by Profs. ZERR, MUKHOPADHYAY, *and others.*



Let $\quad BC = x,\ \angle BCA = \beta$.
Then, for the sphere, we have
$d^2x/dt^2 = 5g/7\sin\beta,\quad x = \frac{5}{14}g\sin\beta\,.\,t^2$......(1).
For the cylinder, we have
$d^2x/dt^2 = 2g/3\cos\beta$,
since $\quad \sin ABC = \cos ACB$,
$x = g/3\cos\beta\,.\,t^2$..................(2),
(1) ÷ (2) gives $\tan\beta = \frac{14}{15}$, $\beta = 43^\circ\ 1'\ 30''+$.

11845. (J. Macleod.)—BC, CA are chords in a circle, the angle BCA being right, and E the point of bisection of CA. BE is produced to meet the circumference in F, and BH is taken on BF, = EF, while a point G on the circumference is taken such that HG = CE. Prove that FG passes through the centre of the circle.

Solution by W. J. Constable, M.A.; W. J. Dobbs, B.A.; *and others.*



Produce GH to meet the circle in X; then

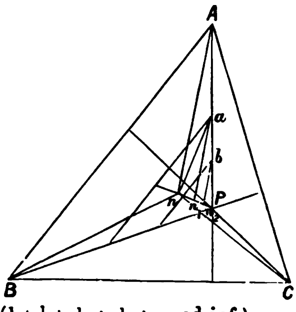
$$GH.HX = BH.HF = BE.EF = AE.EC;$$

$\therefore$ $XH = AE = EC = HG$; hence H is mid-point of the chord XG, and BF passes through mid-point of two equal chords AC, GX; wherefrom it follows that, as BA is a diameter, so also is FG.

11743. (C. Morgan, M.A., R.N.)—Let x, y, z be the distances of the centre of the nine-point circle of $\triangle$ABC from the angular points, d its distance from P the orthocentre. Let the nine-point circle of the triangle formed by joining the middle points of PA, PB, PC be taken, and so on. Prove that, if ρ is the radius of the nine-point circle,

$$2\Sigma(x^2+y^2+z^2) = \tfrac{8}{3}(x^2+y^2+z^2) = \tfrac{4}{3}(PA^2+PB^2+PC^2)+16\rho^2-8d^2.$$

Solution by Professors Zerr, Bhattacharya, *and others.*



Let n be the nine-point centre of ABC, n_1 that of the triangle formed by joining the mid-points of PA, PB, PC, n_2 that of second triangle so formed, and so on. Then $An = x$, $Bn = y$, $Cn = z$, $nP = d$, $na = \rho$.

From the similar triangles

$$AnP,\quad an_1P,\quad bn_2P,\quad \&c.,$$

we get $\quad an_1 = \tfrac{1}{2}An = \tfrac{1}{2}x$;

$$\therefore\quad an_1^2 = \tfrac{1}{4}x^2,$$

$$bn_2 = \tfrac{1}{2}an_1 = \tfrac{1}{4}An = \tfrac{1}{4}x;$$

$$\therefore\quad bn_2^2 = \tfrac{1}{16}x^2.$$

Hence $2\Sigma(x^2+y^2+z^2) = 2(x^2+y^2+z^2)(1+\tfrac{1}{4}+\tfrac{1}{16}+\tfrac{1}{64}+\dots \text{ ad inf.})$
$= \tfrac{8}{3}(x^2+y^2+z^2)$.

For, since $1/(1-x) = 1+x+x^2+x^3+\&c.$, when we make $x = \tfrac{1}{4}$,

we have $\frac{4}{3} = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \&c.$

$$x^2 = AP^2 + d^2 - 2AP_1 d \cos APn, \quad \rho^2 = \tfrac{1}{4}AP^2 + d^2 - APd \cos APn;$$

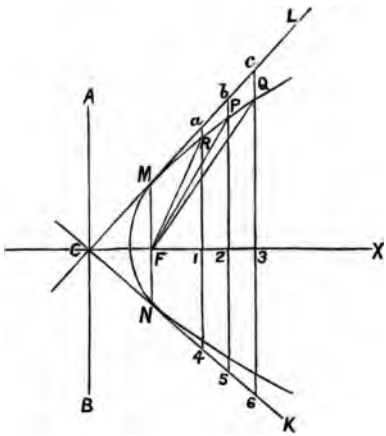
$$\therefore \quad x^2 = \tfrac{1}{2}AP^2 + 2\rho^2 - d^2;$$

$$\therefore \quad x^2 + y^2 + z^2 = \tfrac{1}{2}(AP^2 + BP^2 + CP^2) + 6\rho^2 - 3d^2;$$

whence the result.

9866. (Rev. T. Roach, M.A.)—The ordinate MP of an hyperbola is produced to Q, so that MQ = SP. Find the locus of Q by a geometrical construction. [This theorem is proved analytically in Todhunter's *Conics*, Chap. XI., Ex. 2.]

Solution by Profs. Zerr, Chakrivarti, *and others.*



To construct a conic section, let F be the given point, AB the given straight line, and M : N the given ratio.

Through F draw MN parallel and CX perpendicular to AB. Take FM = FN, so that

FM or FN : FC :: M : N.

Through M, N draw CL and CK; also draw the parallels to MN, $a1$, $b2$, $c3$, &c. Take the length of any perpendicular, as $a1$, on the compasses, and with one foot on F, note where the other falls on $a1$. The points thus found will indicate the curve.

Therefore, by construction, FR = (SP in the example) = $a1$ = (MQ in the example).

Therefore the locus of Q is the straight line CL.

2015. (C. Leudesdorf, M.A.) — If $A = bc - f^2$, $B = ca - g^2$,
$C = ab - h^2$, $F = gh - af$, $G = hf - bg$, $H = fg - ch$,
and if A′, B′, C′, F′, G′, H′ denote similar expressions with regard to the accented letters a', b', c', f', g', h', prove that

$$\begin{vmatrix} hf' + h'f - bg' - b'g, & gh' + g'h - af' - a'f, & ab' + ba' - 2hh' \\ G, & F, & C \\ G', & F', & C' \end{vmatrix} = \begin{vmatrix} a'G + h'F + g'C, & h'G + b'F + f'C \\ aG' + hF' + gC', & hG' + bF' + fC' \end{vmatrix}.$$

Solution by H. J. WOODALL, A.R.C.S.; Professor SARKAR; *and others.*

Take coefficients of C on both sides.

This is $\begin{vmatrix} gh'+g'h-af'-a'f, & hf'+h'f-bg'-b'g \\ F', & G' \end{vmatrix}$

$$-\begin{vmatrix} g', & f' \\ aG'+hF'+gC', & hG'+bF'+fC' \end{vmatrix}$$

$$= G'(gh'-a'f)-F'(h'f-b'g)-C'(fg'-f'g)$$

$$= g(h'G'+b'F'+f'C')-f(a'G'+h'F'+g'C') = 0;$$

so coefficients of F and G are each equal zero;

$\therefore$ the two determinants are equal.

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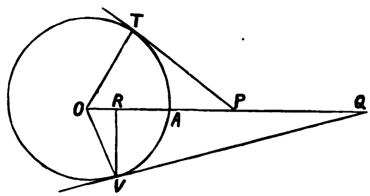
483. (J. H. SWALE.)—In the radius OA, produced, of a circle, take any point P, and draw the tangent PT; produce OP to Q, making PQ = PT, and draw the tangent QV; if VR be drawn perpendicular to OA, prove that PR = PQ = PT.

Solution by J. C. ST. CLAIR.



Join OT, OV; then

$OT^2 = OP^2 - PT^2$

$= (OP+PT)(OP-PT);$

$OV^2 = OQ \cdot OR$

$= (OP+PQ)(OP-PR).$

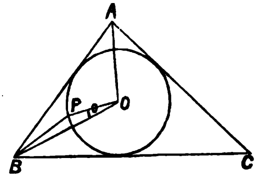
But $OT = OV,$

and $PT = PQ;$

$\therefore \quad PR = PT = PQ.$

11875. (W. J. GREENSTREET, M.A.)—If $\lambda_a, \lambda_b, \lambda_c$ be the joins of any point on the in-circle to the angular points of a triangle, find the value of $\Sigma a\lambda_a^2$.

Solution by C. MORGAN, M.A.; W. J. DOBBS, B.A.; *and others.*



Let $POB = \theta$; then

$POC = 90+\frac{1}{2}A+\theta, \quad POA = 90+\frac{1}{2}C-\theta,$

r = radius of in-circle;

then we have the following equations:—

$PB^2 = r^2+OB^2-2rOB\cos\theta$

$= r^2\left\{1+\operatorname{cosec}^2\frac{1}{2}B-2\operatorname{cosec}\frac{1}{2}B\cos\theta\right\}$

$PC^2 = r^2+OC^2-2rOC\cos(90+\theta+\frac{1}{2}A)$

$= r^2\left\{1+\operatorname{cosec}^2\frac{1}{2}C\right.$

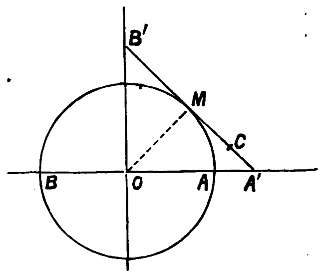
$\left.+2\operatorname{cosec}\frac{1}{2}C\sin(\theta+\frac{1}{2}A)\right\}$

$$PA^2 = r^2 + OA^2 - 2rOA\cos(90 + \tfrac{1}{2}C - \theta)$$
$$= r^2\left\{1 + \operatorname{cosec}^2\tfrac{1}{2}A + 2\operatorname{cosec}\tfrac{1}{2}A\sin(\tfrac{1}{2}C - \theta)\right\};$$
$$\therefore \quad \Sigma a\lambda_a{}^2 = r^2\Sigma a\,(1 + \operatorname{cosec}^2\tfrac{1}{2}A),$$

for the coefficients of $\sin\theta$ and $\cos\theta$ are easily seen to each vanish.

11960. (Professor IGNACIO BEYENS.) — Soit un cercle de diamètre AB; si l'on même un rayon quelconque OM et la droite MA′ qui forme avec AB l'angle MA′O = MOA′, si l'on prend la distance A′C = l (constante) sur A′M, on demande le lieu des points C ainsi déterminés?

Solution by Professor DROZ FARNY; R. KNOWLES, B.A.; *and others.*



Prolongeons MA′ jusqu'à son point de rencontre B′ avec le diamètre perpendiculaire à AB. On a évidemment OM = MB′, et par conséquent A′B′ = 2R.

Le lieu de C est donc une ellipse concentrique au cercle ayant pour demi axes situés sur

OA et OB′ : $2R - l$ et l.

11936. (Professor DE LONGCHAMPS.)—Eliminer le paramètre ϕ entre les deux égalités
$$2x\sin^3\theta = c\sin^2(\phi - \tfrac{1}{2}\theta)\sin(\phi + \tfrac{1}{2}\theta),$$
$$2y\sin^3\theta = c\sin^2(\phi + \tfrac{1}{2}\theta)\sin(\phi - \tfrac{1}{2}\theta).$$

Solution by H. W. CURJEL, B.A.; W. J. DOBBS, B.A.; *and others.*

$$\frac{\sin(\phi - \frac{1}{2}\theta)}{x} = \frac{\sin(\phi + \frac{1}{2}\theta)}{y} = \frac{2\sin\phi\cos\frac{1}{2}\theta}{x+y} = \frac{2\cos\phi\sin\frac{1}{2}\theta}{y-x};$$

therefore $$\tan\phi = \frac{x+y}{y-x}\tan\tfrac{1}{2}\theta.$$

Also $$32xy\sin^6\theta = c^2(\cos\theta - \cos 2\phi)^3$$

$$= c^2\left\{\cos\theta - \frac{1 - \left(\frac{x+y}{y-x}\right)^2\tan^2\frac{1}{2}\theta}{1 + \left(\frac{x+y}{y-x}\right)^2\tan^2\frac{1}{2}\theta}\right\}^3 = c^2\left(\frac{2xy\sin^2\theta}{x^2+y^2-2xy\cos\theta}\right);$$

therefore $$4(x^2+y^2-2xy\cos\theta)^3 = c^2x^2y^2.$$

11772. (W. J. GREENSTREET, M.A.)—Two planets move in circles round the Sun; show that the aberration of one seen from the other will be less in conjunction than in opposition, in the ratio

$$(\sqrt{R}-\sqrt{r})/(\sqrt{R}+\sqrt{r}),$$

when R, r are the radii of the circles.

Solution by H. W. CURJEL, B.A.; C. BICKERDIKE; *and others.*

Let V and v be the velocities of the planets; then

$$\frac{v^2}{r}\times r^2 = \frac{V^2}{R}\times R^2 = \text{attraction of Sun on unit mass at unit distance}$$
$$= c, \text{ say};$$

therefore aberration in conjunction $= k(v-V)$, where k is a constant depending on the velocity of light. Aberration in opposition $= k(V+v)$; therefore aberration in conjunction is less, in the ratio

$$\frac{v-V}{V+v} = \frac{\sqrt{(c/r)}-\sqrt{(c/R)}}{\sqrt{(c/R)}+\sqrt{(c/r)}} = \frac{\sqrt{R}-\sqrt{r}}{\sqrt{R}+\sqrt{r}}.$$

10171. (Col. H. W. L. HIME.)—If α, β, γ are the vectors of the vertices of a triangle ABC, M the mass-centre, E the incentre, I an excentre, P the orthocentre, Q the circumcentre, N the mid-centre (*i.e.*, of nine-point circle); prove (1) that $OM = \frac{1}{3}(\alpha+\beta+\gamma)$,

$$OE = \frac{a\alpha+b\beta+c\gamma}{a+b+c},\quad OI = \frac{\pm a\alpha\pm b\beta\pm c\gamma}{\pm a\pm b\pm c},\quad OP = \frac{\tan A\,\alpha+\tan B\,\beta+\tan C\,\gamma}{\tan A+\tan B+\tan C},$$

$$OQ = \frac{(\tan B+\tan C)\,\alpha+\ldots}{2(\tan A+\tan B+\tan C)},\quad ON = \frac{(2\tan A+\tan B+\tan C)\,\alpha+\ldots}{4(\tan A+\tan B+\tan C)};$$

(2) since $OP+OQ = 2ON$, that N bisects PQ; (3) if X is the intersection of three lines drawn from the vertices, cutting the opposite sides as the xth powers of the other sides, that is, if AX cuts BC in A′, and that

$$A'B : A'C = c^x : b^x,\quad \text{then}\quad OX = \frac{a^x\alpha+b^x\beta+c^x\gamma}{a^x+b^x+c^x},$$

OM and OI being special cases for $x = 0$ and $x = 1$.

Solution by H. J. WOODALL, A.R.C.S.

A more general theorem is this:—If the sides of a triangle be divided in such ratios as $k:l$, $l:m$, $m:k$, respectively, and in order, then OX (X being intersection of the joins of these points to the opposite vertices, by CEVAS' theorem), $OX = (\alpha k+\beta l+\gamma m)/(k+l+m)$.

Proved thus: vector $OA' = (\beta l+\gamma m)/(l+m)$, vector $OB' = (\gamma m+\alpha k)/(k+m)$; assume intersection of AA′, BB′ to divide these lines in the ratios $1:x$, $1:y$ respectively, then

$$OX = \{x\alpha(l+m)+l\beta+m\gamma\}/(x+1)(l+m)$$
$$= \{y\beta(k+m)+k\alpha+m\gamma\}/(y+1)(k+m);$$

now equate coefficients of α, β, γ (since these expressions are identical); therefore $\quad x(l+m)/k = l/y(k+m) = m/m \;(=1)$;
therefore $\quad x = k(l+m)$ and $y = l(k+m)$,
whence required vector is as shown. For OM, $k = l = m$; OE, $k, l, m = a, b, c$; OI, $k, l, m = \pm(a, b, c)$; OP, $k, l, m = \tan A, \tan B, \tan C$ (these are obtained by inspection). The remaining cases are not so easily obtainable. OQ can be obtained simply from a figure and a little work, but ON requires some thought [$2ON = OP + OQ$ not being available (*vide* enunciation)].

11767. (R. Tucker, M.A.)—ABC is a triangle of which DE, FG, HK are equipotential antiparallels; ρ_1, ρ_2, ρ_3 are the radii of (ADE), (BFG), (CHK). Prove that (1) $AD.BF.CH = AE.BG.CK = DE.FG.HK$; (2) $\rho_1 : \rho_2 : \rho_3 = a : b : c$; (3) $\Pi(\Delta\alpha BC) : \Pi(\Delta\alpha DE) = R^6 : \rho_1^2\rho_2^2\rho_3^2$; where α, β, γ are the points of section of the circum-circle ABC by the circles ADE, BFG, CHK.

[By *equipotential antiparallels* are meant antiparallels so drawn that the potency of A with regard to (BCDE) = potency of B with regard to (CFGA) = potency of C with regard to (AHKB). Another property of the points α, β, γ was given in Quest. 4630, in March, 1875.]

Solution by the Proposer.

Let the potency be denoted by μ^2;

then
$$AB.AD = \mu^2 = AC.AE,$$
$$BC.BF = \mu^2 = BA.BG,$$
$$CH.CA = \mu^2 = CB.CK;$$

$$\therefore\; AD.BF.CH = AE.BG.CK = DE.FG.HK$$
(by similar triangles) .... (i.).

Again,
$$\mu^2 = AC.AE = (2R\sin B)(2\rho_1 \sin C);$$
$$\therefore\; \rho_1/\sin A = \rho_2/\sin B = \rho_3/\sin C, \text{ hence } \qquad \text{(ii.)}.$$

Now
$$\frac{\Delta\alpha BC}{\Delta\alpha DE} = \frac{\alpha B.\alpha C}{\alpha D.\alpha E} = \frac{R^2}{\rho_1^2}, \text{ hence } \qquad \text{(iii.)}.$$

9823. (W. J. C. Sharp, M.A.)—Prove that
$$\int \frac{d\theta}{(a\cos^2\theta + b\sin^2\theta)^2} = \frac{a+b}{2a^{\frac{3}{2}}b^{\frac{3}{2}}}\tan^{-1}\left\{\left(\frac{b}{a}\right)^{\frac{1}{2}}\tan\theta\right\} - \frac{a-b}{2ab}\,\frac{\tan\theta}{a+b\tan^2\theta};$$
$$\int x^{m-1}\log x.X^p dx = \frac{x^m(m\log x - 1)}{m^2}X^p - \frac{bnp}{m^2}\int x^{m+n-1}(m\log x - 1)X^{p-1}dx$$

if $$X \equiv a + bx^n;$$

$$\int_0^{\frac{1}{2}\pi} \cos_m \phi \sin^n \phi \, d\phi = \tfrac{1}{2} \left\{\Gamma\left[\tfrac{1}{2}(m+1)\right]\right\} \Gamma\left[\tfrac{1}{2}(n+1)\right] / \left\{\Gamma\left[\tfrac{1}{2}(m+n)+1\right]\right\}.$$

Solution by Professor G. B. M. Zerr.

$$(1) \quad \int \frac{d\theta}{(a\cos^2\theta + b\sin^2\theta)^2} = \frac{1}{a^{\frac{3}{2}}b^{\frac{3}{2}}} \int \cos^2\phi \, (b + a\tan^2\phi) \, d\phi$$

$$= \frac{(a+b)\,\phi}{2a^{\frac{3}{2}}b^{\frac{3}{2}}} - \frac{(a-b)\sin\phi\cos\phi}{2a^{\frac{3}{2}}b^{\frac{3}{2}}},$$

where $$\tan\phi = (a^{-1}b)^{\frac{1}{2}} \tan\theta;$$

substituting $\tan^{-1}\left\{(a^{-1}b)^{\frac{1}{2}}\tan\theta\right\}$ for ϕ, and $\left\{(ab)^{\frac{1}{2}}\tan\theta\right\} / (a + b\tan^2\theta)$ for $\sin\phi\cos\phi$, we get the stated result.

$$(2) \quad \int x^{m-1}\log x \, dx = \int dv \quad \text{or} \quad v = x^m (m\log x - 1)\, m^{-2};$$

$$u = X^p \quad \text{or} \quad du = bnp\, x^{n-1} X^{p-1} dx;$$

substituting in the formula $\int u\,dv = uv - \int v\,du$,

we get the stated result.

[If $n = p-1$ and $m = q-1$, (3) is solved in Art. 122, page 163, of Williamson's *Integral Calculus*.]

11970. (Professor Barisien.)—Si l'on considère une strophoïde dont le nœud est le point O, deux transversales parallèles et symétriques par rapport à O coupent cette courbe respectivement en des points A, B, C, A′, B′, C′ tels que OA . OB . OC = OA′. OB′. OC′.

Solution by Professor Droz Farny, Morel, *and others.*

Equation de la strophoïde $r = \dfrac{a\cos 2\phi}{\cos\phi}$. Equation d'une droite

$$r = \frac{p}{\cos(\phi - \alpha)};$$

p est la distance de O à la droite et α l'inclinaison de p sur l'axe. Eliminons ϕ entre les deux équations. On a

$$\cos(\phi-\alpha) = p/r, \quad \sin(\phi-\alpha) = 1/r \,.\, \sqrt{(p^2-r^2)},$$

de là, $$\cos\phi = p/r \,.\, \cos\alpha - (\sin\alpha)/r \,.\, \sqrt{(p^2-r^2)},$$

et $$\sin\phi = p/r \,.\, \sin\alpha + (\cos\alpha)/r \,.\, \sqrt{(p^2-r^2)}.$$

En portant ces valeurs dans la première équation, on obtient

$$r^6 \sin^2 a - r^4 (\dots) + r^2 (\dots) - (4p^4a^2 \cos^2 2a + 4p^4a^2 \sin^2 2a) = 0,$$

$$OA \,.\, OB \,.\, OC = 2p^2a \operatorname{cosec} a.$$

Or cette valeur ne change pas si on remplace p par $-p$ et par conséquent

$$OA \,.\, OB \,.\, OC = OA' \,.\, OB' \,.\, OC' = \frac{2p^2a}{\sin a}.$$

11806. (Morgan Brierley.)—Show how the greatest possible ellipse may be inscribed in a given segment of a circle.

Solution by Professors Zerr, Mukhopadhyay, *and others.*

Let $a^2y^2 + b^2x^2 = a^2b^2$ be the equation to the ellipse; then

$$x^2 + (y + b + c)^2 = r^2$$

is the equation to circle, where c is the distance of the base of the segment from the centre; $\therefore\ a^2y^2 - b^2(y+b+c)^2 + b^2r^2 - a^2b^2 = 0$ has equal roots;

$$\therefore\ b^4(b+c)^2 = (a^2-b^2)\, b^2 \left\{r^2 - a^2 - (b+c)^2\right\},$$

or $$b = a^2r^{-2}c \pm ar^{-2}\left\{(r^2-c^2)(r^2-a^2)\right\}^{\frac{1}{2}} \quad \dots\dots\dots\dots\dots \text{(A)};$$

but $\pi ab = \text{maximum}$; $\therefore\ u = ab = a^3r^{-2}c \pm a^2r^{-2}\left\{(r^2-c^2)(r^2-a^2)\right\}^{\frac{1}{2}}$.

After differentiating, we get $a = 0$, a line ellipse, or

$$a = \left\{\tfrac{1}{8}\left[4r^2 - c^2 \pm c\sqrt{(8r^2+c^2)}\right]\right\}^{\frac{1}{2}};$$

the positive sign gives a for the segment larger than a semicircle, the negative for the one smaller. This value of a in (A) gives the two corresponding values of b.

9956. (B. F. Finkel.)—Find the volume removed by boring an inch hole diagonally through a 10-inch cube.

Solution by Professor G. B. M. Zerr.

This evidently means that the centre of the cross section of the auger is always on the diagonal of the cube.

Let r = the radius of auger, b = side of cube. Now, when the circumference of a cross-section of the auger just intersects the edges of the cube, we have, for the part cut away, a triangular pyramid, of altitude $\frac{1}{2}r\sqrt{2}$ and side of base $r\sqrt{3}$;

$$\therefore\ \text{volume of two such pyramids} = 2 \times \tfrac{3}{4}r^2\sqrt{3} \times \tfrac{1}{6}r\sqrt{2} = \tfrac{1}{4}r^3\sqrt{6},$$

$$\text{diagonal of cube} = b\sqrt{3},$$

length of diagonal between the bases of the two pyramids is $b\sqrt{3}-r\sqrt{2}$
$= \text{length of cylinder}, \quad \text{volume of cylinder} = \pi r^2(b\sqrt{3}-r\sqrt{2}),$
volume of the six ungulas is $6(\frac{3}{8}r^3\sqrt{3}-\frac{1}{8}\pi r^3)\times\frac{1}{2}r\sqrt{2}/\frac{1}{2}r = r^3(\frac{9}{4}\sqrt{6}-\pi\sqrt{2})$.

$$\text{Volume removed} = \pi r^2(b\sqrt{3}-r\sqrt{2})+\tfrac{1}{4}r^3\sqrt{6}-r^3(\tfrac{9}{4}\sqrt{6}-\pi\sqrt{2})$$
$$= r^2(\pi b\sqrt{3}-2r\sqrt{6}).$$

But $r=\frac{1}{2}$, $b=10$; $\therefore$ $V=\frac{1}{4}\sqrt{3}(10\pi-\sqrt{2}) = 12{\cdot}991122664$ cu. in.

10658. (Professor SVÉCHNIKOFF.)—Résoudre les équations

$$x+y+z+t=4m, \quad x^2+y^2+z^2+t^2=4m^2+4q^2,$$
$$x^3+y^3+z^3+t^3=4m^3+12mq^2, \quad x^4+y^4+z^4+t^4=4m^4+24m^2q^2+4q^4+4p^4.$$

Solution by H. J. WOODALL, A.R.C.S.; Professor ZERR; *and others.*

Let x, y, z, t be the roots of the equation $x^4+ax^3+bx^2+cx+d=0$. Then, by means of Newton's equations, viz.:— $S_1+a=0$,
$S_2+S_1a+2b=0, \quad S_3+S_2a+S_1b+3c=0, \quad S_4+S_3a+S_2b+S_1c+4d=0,$
and the given values of S_1, S_2, S_3, S_4, we find

$$a=-4m, \quad b=6m^2-2q^2, \quad c=4mq^2-4m^3, \quad d=m^4+2m^2q^2+q^4-p^4,$$

and equation is

$$x^4-4mx^3+(6m^2-2q^2)x^2+(4mq^2-4m^3)x+m^4+2m^2q^2+q^4-p^4=0;$$
$$\therefore\ (x^2-2mx+m^2-q^2)^2=p^4; \quad \therefore\ (x-m)^2=q^2\pm p^2;$$
$$\therefore\ x=m\pm(q^2\pm p^2)^{\frac{1}{2}}; \quad \therefore\ x, y, z, t=m\pm(q^2\pm p^2)^{\frac{1}{2}}.$$

11961. (Professor NEUBERG.)—Etant donné un tétraèdre ABCD, trouver un point P tel que les plans menés par P parallèlement à une face coupent les trièdres opposés suivant des triangles équivalents. Chercher la surface de ces triangles.

Solution by H. J. WOODALL, A.R.C.S.

Let the areas of the faces be A, B, C, D, the corresponding heights a, b, c, d; then, if P be (x, y, z, w) and Δ = solidity of the tetrahedron, we have

α = required area $=(a-x)A/a=(b-y)B/b=(c-z)C/c=(d-w)D/d$,
which give $\qquad x=a(A-\alpha)/A$, &c.

But $\qquad Ax+By+Cz+Dw=6\Delta=aA=bB=cC=dD.$

Thence we shall easily find $\alpha=18\Delta/(a+b+c+d)$,

$$x=a-3a^2/(a+b+c+d), \quad y=b-3b^2/(a+b+c+d), \text{ \&c.}$$

12020. (Professor SYLVESTER.)—Find the general values of u and v, as rational integral functions of x, which satisfy the equations

$$u^2+l\,(du/dx)^2 = v^2 \dots (1), \qquad u^2+l\,(du/dx)^2 = v^2+\lambda\,(dv/dx)^2 \dots (2).$$

Solution by H. J. WOODALL, A.R.C.S.

Write the equations

$$u^2+l\,(du/dx)^2 = v^2 \dots (1), \qquad u^2+m\,(du/dx)^2 = v^2+\lambda\,(dv/dx)^2 \dots (2).$$

Subtract; therefore $\lambda\,(dv/dx)^2 = (m-l)(du/dx)^2$;

therefore $v = \{(m-l)/\lambda\}^{\frac{1}{2}}\,u+\mathrm{C}$;

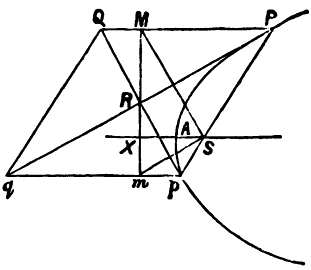
therefore $u^2+l\,(du/dx)^2 = \left[\{(m-l)/\lambda\}^{\frac{1}{2}}\,u+\mathrm{C}\right]^2 = (1+\mathrm{A})\,u^2+2\mathrm{B}u+\mathrm{C}$;

therefore $l\,(du/dx)^2 = \mathrm{A}u^2+2\mathrm{B}u+\mathrm{C}$; $\therefore\ x = l^{\frac{1}{2}}\displaystyle\int\frac{du}{(\mathrm{A}u^2+2\mathrm{B}u+\mathrm{C})^{\frac{1}{2}}}$.

This gives x in terms of u; we can then find u in terms of x, and thence v in terms of x without difficulty.

11997. (Professor GENESE, M.A.)—If PSp be a focal chord of a conic, and PM, pm perpendiculars on the corresponding directrix, prove that parallels through P, p to Sm, SM respectively meet at the middle point of Mm.

Solution by V. J. BOUTON, B.Sc.; H. W. CURJEL, B.A.; *and others.*



Let the lines through P, p parallel respectively to Sm, SM, meet in R. Produce PR to meet pm produced in q; then $\dfrac{\mathrm{SP}}{\mathrm{PM}} = \dfrac{\mathrm{S}p}{pm} = \dfrac{\mathrm{SP}+\mathrm{S}p}{\mathrm{PM}+pm}$;

but $\dfrac{\mathrm{SP}}{mq} = \dfrac{\mathrm{S}p}{pm}$;

therefore $mq = \mathrm{PM}$.

Similarly, if pR meets PM in Q,

$\mathrm{MQ} = pm$.

Therefore PpqQ is a parallelogram, and its diagonals Pq, pQ bisect one another at R.

Since the triangles PRM, qRm are equal in all respects, R must lie on the directrix Mm, and must be the middle point of Mm.

[By Statics, let PS, PM represent two forces; then their resultant bisects SM. But Sp, pm are proportional to the forces and in the same directions; therefore the resultant is parallel to Sm. Hence &c.; the theorem follows by EUCLID, VI. 2.]

12087 & 12107. 12087. (Rev. T. P. KIRKMAN, M.A., F.R.S.)—If ${}_k\mathbf{R}_x$ denote the number of possible permutations of the filled k-partitions of x, *i.e.*, of the partitions ${}_kQ_x$, in which zero and repeated parts are permitted, and if C_k^{x+k-1} be the kth coefficient in $(1+1)^{x+k-1}$, then ${}_k\mathbf{R}_x = C_k^{x+k-1}$.

12107. (RUSTICUS.)—Write down the terms of A whose sum is the kth term of B, in

$$A = (a+b)^{\alpha}(a+b)^{\beta}(a+b)^{\gamma}\ldots = (a+b)^{\alpha+\beta+\gamma+\ldots} = (a+b)^{s} = B.$$

Solution by Rev. T. P. KIRKMAN, M.A., F.R.S.; *and* D. BIDDLE.

1. Let ${}_kQ_x$ be the number of the filled k-partitions of x, *i.e.*, of the partitions in which repeated parts and zero parts are allowed, and let these ${}_kQ_x$ partitions be written in dictionary order, so that no part shall precede a lesser.

Let ${}_kQ_x'$ be the number of all the permutations of the ${}_kQ_x$ partitions that can be added to ${}_kQ_x$. Then it is plain that if

$${}_k\mathbf{R}_x = {}_kQ_x + {}_kQ_x' \qquad\qquad (a),$$

${}_k\mathbf{R}_x$ is the algebraic sum of the permutations of the ${}_kQ_x$ partitions, for all values of x and of k. Let $k=1$, then $Q_x = {}_1Q_x = 1$, for there is only one way of writing the number x in one part only, and ${}_1Q_x' = 0$, for to that one part can be added no permutation of it. Thus, whatever be x, (a) becomes

$${}_1\mathbf{R}_x = {}_1Q_x = 1 \qquad\qquad (b).$$

Theorem Ω.—If ${}_k\mathbf{R}_x$ be the sum of the permutations of the k-partitions of x in which repeated and zero parts are allowed, ${}_k\mathbf{R}_x$ is the kth coefficient in $(1+1)^{x+k-1}$, where $k>x$, or $=x$, or $<x$.

2. Let $C_k^i = C_k^{(i-k+1)+k-1}$ be the kth coefficient in $(1+1)$; and let C^{x+k-1} be the kth coefficient in $(1+1)^{x+k-1}$,

$$C_k^{x+k-1} = \frac{(x+k-1)(x+k-2)\ldots(x+2)(x+1)}{1\,.\,2\ldots(k-2)(k-1)}$$

$$= \frac{k-1+x}{k-1}\cdot\frac{(x+k-2)(x+k-3)\ldots(x+2)(x+1)}{1\,.\,2\ldots(k-3)(k-2)}$$

$$= \frac{(x+k-2)(x+k-3)\ldots(x+2)(x+1)}{1\,.\,2\ldots(k-3)(k-2)} + \frac{(x+k-2)(x+k-3)\ldots(x+1)\,x}{1\,.\,2\ldots(k-2)(k-1)};$$

i.e.,

$$C_k^{x+k-1} = C_{k-1}^{x+k-2} + C_k^{x+k-2} \qquad\qquad (c),$$

where the first member contributes to its equivalent on the right first itself with its two affixes each reduced by unity, and next itself with its upper affix only so reduced.

Making the reduction (c) of every term that appears in the right members, we easily obtain the following additional equivalents of C_k^{x+k-1},

$$\begin{aligned} C_k^{x+k-1} &= C_{k-2}^{x+k-3} + 2C_{k-1}^{x+k-3} + C_k^{x+k-3} \\ &= C_{k-3}^{x+k-4} + 3C_{k-2}^{x+k-4} + 3C_{k-1}^{x+k-4} + C_k^{x+k-4} \\ &= C_{k-4}^{x+k-5} + 4C_{k-3}^{x+k-5} + 6C_{k-2}^{x+k-5} + 4C_{k-1}^{x+k-5} + C_k^{x+k-5} \\ &= C_{k-5}^{x+k-6} + 5C_{k-4}^{x+k-6} + 10C_{k-3}^{x+k-6} + \text{\&c. \&c.} \qquad (d) \end{aligned}$$

The law of these equivalents is evident, and the series is to be continued until an expression is obtained that has been obtained before.

The equivalents (d) are all included in the one following :—

$$C_k^{x+k-1} = C_1^i \cdot C_{k-i}^{x+k-i-1} + C_2^i \cdot C_{k-(i-}^{x+k-i-1} + C_3^i \cdot C_{k-(i-2)}^{x+k-i-1}$$
$$+ C_4 \cdot C_{k-(i-3)}^{x+k-i-1} + C_5^i \cdot C_{k-(i-4)}^{x+k\ i-1} + \&c.\ \&c.\ \ldots\ (e),$$

to be continued by the obvious law, with every value in turn of $i > 0$, until an expression is repeated with the next i.

Every product has the form C_k^J or C_K^j, which vanishes if either $k < 1$ or $K < 1$; for no binomial $(1+1)^j$ has a $(k-k)$th or a $(k-k-1)$th coefficient > 0. It vanishes also if $k > 1 + j$, or if $K > 1 + J$; for no binomial $(1+1)^j$ has a $(j+2)$th or $(j+3)$th coefficient.

There is no limit to the number of the above different expressions by the increasing i, if either x or k be large enough, however small the other, k or x, may be.

3. In ${}_kQ_x$ written in dictionary order there is a column in which every partition begins with ϵ zeros, and a different column for every value of ϵ from $\epsilon = k-1$ down to $\epsilon = 0$.

Calling these columns $A_{k-1}, A_{k-2}, \ldots A_2, A_1, A_0$, the column A_ϵ is

$$\left.\begin{array}{l}
0^\epsilon,\ 1^{k-\epsilon-1},\ (x+\epsilon+1-k) \\
0^\epsilon,\ 1^{k-\epsilon-2},\ 2,\ (x+\epsilon-k) \\
C^\epsilon,\ 1^{k-\epsilon-2},\ 3,\ (x+\epsilon-k-1) \\
\vdots \\
0^\epsilon,\ 1^a 2^b 3^c \ldots (x-a-2b-3c-\ldots) \\
\vdots \\
0^\epsilon,\ r^a (r+1)^b (r+2)^c \ldots \left[x-ar-b(r+1)-c(r+2)-\ldots\right]
\end{array}\right\} \ldots (A).$$

The final part of every partition is x, reduced by the sum of the $k-1$ preceding parts.

The indices, all $\overline{>}\ 0$, being repetition-numbers of the parts carrying them, are to be read as mere multipliers.

When $\epsilon = k-1$, the column A_ϵ is reduced to its first term,

$$0^{k-1},\ x;\quad = A_{k-1}.$$

If ${}_kQ_x$ is ${}_{10}Q_{26}$, and $\epsilon = 3$, the fourth above written partition may be $0^3 1^2 2^0 3 4^3\ (26-17)$, which is 0001134449.

In the last written partition, r is the greatest integer in $x : (k-\epsilon)$. If $x = (k-\epsilon)\,r$, $a = k-\epsilon$, $b = c = 0$, and the partition is 0_ϵ, $r^{k-\epsilon} = 0_\epsilon, r, r, \ldots$ If $x = (k-\epsilon)\,r+1$, or $= (k-\epsilon)\,r+2$, the last is 0_ϵ, $r^{k-\epsilon-1}$, $(r+1)$, or 0^ϵ, $r^{k-\epsilon-2}$, $(r+1)$, $(r+1)$.

Consider now the two columns

$$\left.\begin{array}{l}
1^{k-\epsilon-1},\ (x+\epsilon+1-k) \\
1^{k-\epsilon-2},\ 2,\ (x+\epsilon-k) \\
1^{k-\epsilon-2},\ 3,\ (x+\epsilon-k-1) \\
\vdots \\
1^a,\ 2^b,\ 3^c,\ \ldots (x-a-2b-3c\ldots) \\
\vdots \\
r^a (r+1)^b \ldots \\
\quad \ldots \left[x-ar-b(r+1)-\ldots\right]
\end{array}\right\} B_\epsilon,\qquad
\left.\begin{array}{l}
0^{k-\epsilon-1},\ (x+\epsilon-k) \\
0^{k-\epsilon-2},\ 1,\ (x+\epsilon-k-1) \\
0^{k-\epsilon-2},\ 2,\ (x+\epsilon-k-2) \\
\vdots \\
0^a,\ 1^b,\ 2^c, \ldots (x-1-a-2b-\ldots) \\
\vdots \\
(r-1)^a,\ r^b,\ \ldots \\
\quad \ldots \left[x-1-ar-b(r+1)-\ldots\right]
\end{array}\right\} b_\epsilon.$$

B_ϵ is A_ϵ without its ϵ vertical rows of zeros, and (b_ϵ) is what (B) becomes by diminishing by unity every part in every $(k-\epsilon)$-partition in (B_ϵ). This (b_ϵ) is now in one column, the whole of ${}_{k-\epsilon}Q_{x-k+\epsilon}$, the $(k-\epsilon)$-partitions of $x-k+\epsilon$, in which are both repeated and zero parts. (B_ϵ) is the whole of ${}_{k-\epsilon}P_x$, the $(k-\epsilon)$-partitions of x in which are repetitions, but no zero parts.

4. We are about to compare the number of the possible permutations of the partitions in the partial column (A_ϵ) with that of those possible in the partitions (b_ϵ) of the complete ${}_{k-\epsilon}Q_{x-k+\epsilon}$; namely with the number ${}_{k-\epsilon}\mathbf{R}_{x-k+\epsilon}$. The summed permutations possible in all the columns of k-partitions $A_{k-1}, A_{k-2}, \ldots A_2, A_1, A_0$ are the number ${}_k\mathbf{R}_x$.

Let ${}^{\epsilon}_{k}\mathbf{R}_x$ be the number of them in (A_ϵ). We are then about to compare the numbers ${}^{\epsilon}_{k}\mathbf{R}_x$ and ${}_{k-\epsilon}\mathbf{R}_{x-k+\epsilon}$ of permutations possible in the columns (A_ϵ) and (b_ϵ). These numbers depend not upon the parts, but upon the indices $\overline{\overline{>}}\ 0$, carried by the parts; *e.g.*, looking at the columns (B_ϵ) and (b_ϵ), we see that the number of permutations of the fourth written $(k-\epsilon)$-partition is the same in both columns, namely, $(k-\epsilon)! \times (a!\,b!\,c!\ldots)^{-1}$, and the like number in the fourth partition in (A_ϵ) is $k!\,(\epsilon!)^{-1}(a!\,b!\,c!\ldots)^{-1}$; where, if an index, say b, is $b=0$, we write, as usual, 1 for $0!$, and $(a!\,c!)^{-1}$ for $(a!\,b!\,c!\ldots)^{-1}$.

Exactly the like is true of the permutations of the mth partition in the columns (A_ϵ) and (b_ϵ) that we are comparing. Both have the same number q of partitions. If that mth partition of (A_ϵ) writes under $k!\,(\epsilon!)^{-1}$ the denominator D_m, the same D_m is written by the mth partition of (b_ϵ) under $(k-\epsilon)!$, as, in the case of that fourth partition, was done by both columns; the same D_4 was written by both.

Hence if (A_ϵ) gives us for the sum of all its permutations

$$ {}^{\epsilon}_{k}\mathbf{R}_x = k!\,(\epsilon!)^{-1} \times \left\{D_1^{-1}+D_2^{-1}+D_3^{-1}+\ldots+D_q^{-1}\right\}, $$

showing a denominator $D_i \overline{\overline{>}}\ 1$ for the ith partition, the column (b_ϵ) will give us for its sum,

$$ {}_{k-\epsilon}\mathbf{R}_{x-k+\epsilon} = (k-\epsilon!) \times \left\{D_1^{-1}+D_2^{-1}+D_3^{-1}+\ldots+D_q^{-1}\right\}, $$

showing the same D_i as denominator for the ith partition, whence follows

$$ {}^{\epsilon}_{k}\mathbf{R}_x = \frac{k!}{\epsilon!\,(k-\epsilon)!}\,{}_{k-\epsilon}\mathbf{R}_{x-k+\epsilon} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (f); $$

and this for every value of ϵ from $\epsilon = k-1$ down to $\epsilon = 0$.

Lemma.—Let $x+k=n$, a given number, and let it be given, as known by trial or otherwise, that Theorem Ω is true for all values of x and k that satisfy $x+k<n$.

Since, because $x-k+\epsilon+k-\epsilon<x+k$ and $=x$, ${}_{k-\epsilon}\mathbf{R}_{x-k+\epsilon}$ is the $(k-\epsilon)$th coefficient of $(1+1)^{x-1}$ (by Theorem Ω), which is $C^{x-1}_{k-\epsilon}$, by Art. 2, and since the first factor on the right in (f) is the $(k-\epsilon+1)$th coefficient of $(1+1)^k$, which is $C^k_{k-\epsilon+1}$, by Art. 2, (f) is

$$ \mathbf{R}_x = C^k_{k-\epsilon+1} \times C^{x-1}_{k-\epsilon}. $$

${}_k\mathbf{R}_x$, the sum of ${}^{\epsilon}_{k}\mathbf{R}_x$ (Art. 4) for every ϵ, is

$$ {}_k\mathbf{R}_x = \mathrm{S}^{\epsilon=0}_{\epsilon=k-1}\left\{\mathrm{C}^{k}_{k-\epsilon+1}\times\mathrm{C}^{x-1}_{k-\epsilon}\right\} $$
$$ = \mathrm{C}^{k}_{2}\cdot\mathrm{C}^{x-1}_{1}+\mathrm{C}^{k}_{3}\cdot\mathrm{C}^{x-1}_{2}+\mathrm{C}^{k}_{4}\cdot\mathrm{C}^{x-1}_{3}+\mathrm{C}^{k}_{5}\cdot\mathrm{C}^{x-1}_{4}+\text{\&c. \&c.}, $$

to be continued so long as the terms have value; see the conditions in Art. 2.

If in (e) in Art. 2, we put $i = k$, we get

$$ \mathrm{C}^{x+k-1}_{k} = 0+\mathrm{C}^{k}_{2}\cdot\mathrm{C}^{x-1}_{1}+\mathrm{C}^{k}_{3}\cdot\mathrm{C}^{x-1}_{2}+\mathrm{C}^{k}_{4}\cdot\mathrm{C}^{x-1}_{3}+\mathrm{C}^{k}_{5}\cdot\mathrm{C}^{x-1}_{4}+\text{\&c., \&c.}, $$

exactly the last written value. It is thus proved that for $x+k=n$ (*vide* Lemma) ${}_k\mathbf{R}_x = \mathrm{C}^{x+k-1}_{k}$, which (Art. 2) is the kth coefficient in $(1+1)^{x+k-1}$; that is, Theorem Ω is true (the Lemma granted) for all values of x and k that satisfy $x+k<n+1$ ....(g), n being any given number. Can we now demonstrate what is begged in the Lemma?

Let $n = 4$: the values of x and k, whose sum < 4, are $x = 1$, $k = 1$; $x = 1$, $k = 2$; $x = 2$, $k = 1$. To complete our demonstration we have only to prove without assumption that

$$ {}_1\mathbf{R}_1 = \mathrm{C}^{1+1-1}_{1},\ {}_1\mathbf{R}_2 = \mathrm{C}^{2+1-1}_{1},\ {}_2\mathbf{R}_1 = \mathrm{C}^{1+2-1}_{2} \ldots\ldots (h,\ h_1,\ h_2). $$

By (b), Art. 1, it is proved that the first members of (h) and of (h_1), are each unity; and their second members, being the first coefficients of the binomials $(1+1)^1$ and $(1+1)^2$, are also each unity. That is, (h) and (h_1) are both true.

For (h_2), the first member is, by definition in Theorem Ω, the sum of the permutations of ${}_2\mathrm{Q}_1$ (Art. 1). The only 2-partition of 1 is 01, which has no permutation but 10. Thus ${}_2\mathbf{R}_1 = 2$ in (h_2).

The second member of (h_2) is the second coefficient (Art. 2) of $(1+1)^2$. Thus $\mathrm{C}^{2}_{2} = 2$ also; and (h_2), (h_1) and (h) are all proved true. Our Lemma has now become the demonstrated truth that Theorem Ω is true for all values of x and k that satisfy $x+k<4$, ($n = 4$); for which can be written by (g)—that satisfy $x+k<5$, ($n = 5$); for which can be written by (g)—that satisfy $x+k<6$, ($n = 6$), &c. &c., up to any value of n. Thus, Theorem Ω is demonstrated for every two finite numbers x and k.

Its value is already proved, for it has done nearly all the work in the solution of the very difficult general problem of the $(k+1)$-partitions of the R-gon *Vide* "The Manchester Memoirs," 1893–4.

The solution of Quest. 12107 having less reference to coefficients, proceeds on somewhat different lines from the above, but derives great aid from the theorem therein propounded.

The indices of a and b respectively, in the kth term of $(a+b)^{\epsilon}$, are $\epsilon-k+1$ and $k-1$. Consequently the terms of $(a+b)^{\alpha}$, $(a+b)^{\beta}$, $(a+b)^{\gamma}$, &c., that enter into the formation of such kth term of $(a+b)^{\epsilon}$, must be those in which the indices of a and b do not exceed their respective limits, $\epsilon-k+1$ and $k-1$. Where α, or β, or γ, or δ, &c. = or < $k-1$, and also = or < $\epsilon-k+1$, all the terms of the expanded function comprised under it enter into the formation of the kth term of $(a+b)^{\epsilon}$. Where

α, or β, or γ, or δ, &c. $> \epsilon - k + 1$, the commencing term of the particular expanded function is given by $(\alpha, \beta, \gamma, \delta, \text{\&c.}) + 1 - (\epsilon - k + 1)$; and where α, or β, or γ, or δ, &c. $> k - 1$, the last available term of the particular expanded function is given by $(\alpha, \beta, \gamma, \delta, \text{\&c.}) - (k-1)$. Having thus separated the available terms of the several minor expansions, the above solution shows how the successive factors can be written down in dictionary order, until each index of the kth term of B is filled up. Thus, let $\alpha = 2$, $\beta = 3$, $\gamma = 4$, $\epsilon = 9$, and let it be required to write down the terms of A which form the eighth term of $(a+b)^9$, which is $36a^2b^7$. The following are the minor expansions, with their available terms marked off:—

$$| a^2 + 2ab + b^2 | \; ; \quad a^3 + | 3a^2b + 3ab^2 + b^3 | \; ; \quad a^4 + 4a^3b + | 6a^2b^2 + 4ab^3 + b^4 | \, .$$

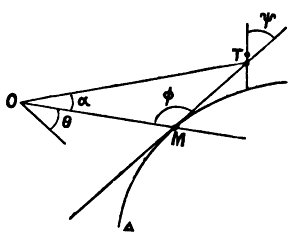
We then have

$$\begin{aligned} a^2 \,.\, b^3 \,.\, b^4 &= a^2b^7 \\ 2ab \,.\, 3ab^2 \,.\, b^4 &= 6a^2b^7 \\ 2ab \,.\, b^3 \,.\, 4ab^3 &= 8a^2b^7 \\ b^2 \,.\, 3a^2b \,.\, b^4 &= 3a^2b^7 \\ b^2 \,.\, 3ab^2 \,.\, 4ab^3 &= 12a^2b^7 \\ b^2 \,.\, b^3 \,.\, 6a^2b^2 &= \underline{6a^2b^7} \\ & 36a^2b^7 \end{aligned}$$

To do this, there is no need to have the minor expansions before our eyes.

11992. (Professor Neuberg.)—On donne une courbe plane Δ. En un point quelconque M de Δ, on mène la tangente MT. O étant un point fixe, on construit l'angle MOT égal à un angle donné α. Trouver la tangente en T à la courbe décrite par ce point lorsque M parcourt Δ.

Solution by H. J. Woodall, A.R.C.S.; Prof. Chakrivarti; *and others.*



Transfer the origin of polar coordinates to point O (any initial line). Then $\text{OM} = r$, $\text{OMT} = \phi$,

$$\text{OT} = r \sin\phi \operatorname{cosec}(\phi + \alpha),$$
$$-\tan\phi = r\, d\theta/dr,$$
$$\sin\phi = r\, d\theta/dr \left\{1 + (r\, d\theta/dr)^2\right\}^{\frac{1}{2}},$$
$$\text{R} = \text{OT} = r\sin\phi \operatorname{cosec}(\phi+\alpha) = r^2 / (r\cos\alpha + dr/d\theta \sin\alpha),$$
$$-\tan\psi = \text{R}\,\frac{d(\theta+\alpha)}{d\text{R}}.$$

This may be obtained when the preceding formulæ have been reduced.

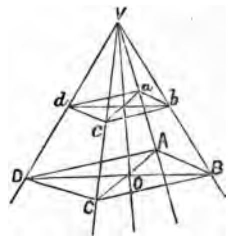
If $\quad r = f(\theta), \quad \text{R} = \{f(\theta)\}^2 / \{\cos\alpha f(\theta) + \sin\alpha f'(\theta)\}$,

$$-\tan\psi = \{f(\theta)\}^2 / \{\cos\alpha f(\theta) + \sin\alpha f'(\theta)\} \times d\theta/d\text{R}$$
$$= \frac{f(\theta)\,\{\cos\alpha f(\theta) + \sin\alpha f'(\theta)\}}{f(\theta) f'(\theta)\cos\alpha + 2\{f'(\theta)\}^2 \sin\alpha - \sin\alpha f(\theta) f''(\theta)}.$$

12012. (W. J. GREENSTREET, M.A.)—ABCD is a quadrilateral, O the intersection of its diagonals, V the vertex of the pyramid which has ABCD as its base. Every section of the pyramid perpendicular to VO is found to be a parallelogram. Find the locus of V.

Solution by Prof. DROZ-FARNY; H. W. CURJEL, B.A.; *and others.*

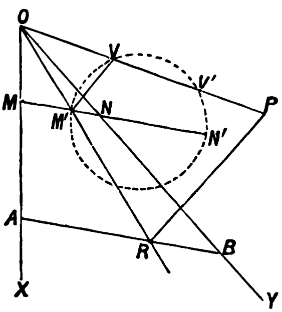
Let *abcd* be a section of the pyramid; then the diagonals of *abcd* cut in VO, but they bisect one another; therefore VO bisects the angles AVC, BVD. Hence locus of V is the circle which is the common section of the spheres on the lines joining O to its harmonic conjugates with respect to A, C and B, D as diameters.



12003. (Professor COUPEAU.)—Étant donnés un point P et un angle XOY, mener, dans une direction donnée, une sécante coupant OX, OY en deux points A, B, tels que le rapport PA : PB ait une valeur donnée $m : n$.

Solution by H. W. PYDDOKE; Prof. GREENWOOD, *and others.*

Draw any line in the given direction, and cutting OX, OY at M, N; and divide MN externally and internally at M′, N′ in the ratio m/n. On M′N′ as diameter describe a circle cutting OP at V, V′. Draw PR parallel to VM′, and meeting OM′ at R. Draw ARB parallel to MN, and to the given direction.



Then the triangles APB, MVN can easily be shown to be similar; therefore

$$AP : PB = MV : VN$$
$$= MM' : M'N = m : n,$$

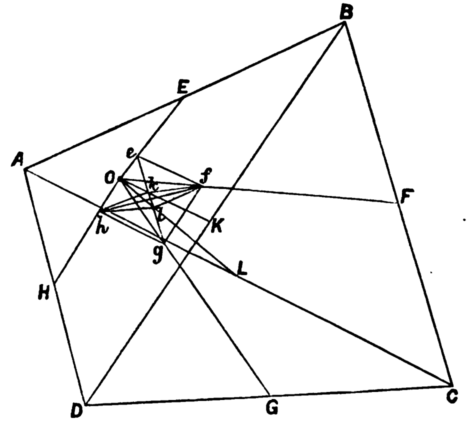
by a well-known property of the harmonically divided line.

If the circle on M′N′ does not cut OP there is no solution. If it does there are two solutions, according as we start from the points V or V′.

8627. (SARAH MARKS.)—ABCD is a quadrilateral, E, F, G, H the middle points of its sides taken in order, K, L the middle points of the diagonals, O any point, OE, OF, OG, OH, OK, OL are divided in the same ratio in e, f, g, h, k, l; prove that eg, fh, kl bisect each other.

Solution by H. J. WOODALL; Prof. BHATTACHARYA; *and others.*

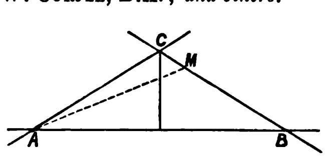
Since $Oe : Of : Og : Oh = OE : OF : OG : OH$, and EFGH is a parallelogram, therefore so is $efgh$; therefore eg, fh bisect each other.



Similarly $elgk$ is a parallelogram; therefore eg, lk, fh bisect each other.

11627. (Professor DURÁN LORIGA.)—Sean a una de las alturas iguales de un triángulo isósceles, l, l_1 los dos segmentos aditivos ó sustractivos en que la perpendicular considerada divide al lado correspondiente (siendo l el contado á partir de la base); demostrar que se verifica la relación $l^2 \pm 2ll_1 = a^2$.

Solution by J. BURKE, M.A.; H. W. CURJEL, B.A.; *and others.*



Sean AC, BC los lados iguales de el triangulo isósceles ABC,

$$AM = a, \quad BM = l, \quad MC = l_1,$$

Entonces $l^2 = AB^2 - a^2$,

$$l_1^2 = BC^2 - a^2;$$

$\therefore \quad l^2 l_1^2 = (AB^2 - a^2)(BC^2 - a^2)$,

pero $\qquad a^2 . BC^2 = AB^2 (BC^2 - \frac{1}{4}AB^2) = 4\Delta^2.$

Siendo Δ la área de ABC; $\quad \therefore \quad \pm ll_1 = a^2 - \frac{1}{2}AB^2$;

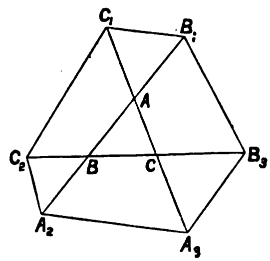
$$\therefore \quad l^2 \pm 2ll_1 = AB^2 - a^2 + 2(a^2 - \tfrac{1}{2}AB^2) = a^2.$$

12027. (Professor NEUBERG.)—On porte sur les côtés BA, CA d'un triangle ABC les longueurs AB_1, AC_1 égales à BC; sur les côtés CB, AB, les longueurs BC_2, BA_2 égales à CA; sur les côtés AC, BC les longueurs CA_3, CB_3 égales à AB. Trouver la surface de l'hexagone $A_2A_3B_3B_1C_1C_2$.

Solution by J. M. STOOPS, B.A.; H. J. WOODALL, A.R.C.S.; *and others.*

$$\Delta AA_2A_3 = \tfrac{1}{2}(b+c)^2 \sin A,$$
$$\Delta AB_1C_1 = \tfrac{1}{2}a^2 \sin A.$$

$$\begin{aligned}\text{Hexagon} &= AA_2A_3 + BB_1B_3 + CC_1C_2 \\ &\quad + AB_1C_1 + BA_2C_2 \\ &\quad + CA_3B_3 - 2ABC \\ &= \tfrac{1}{2}\left\{(b+c)^2 \sin A + (c+a)^2 \sin B \right. \\ &\quad + (a+b)^2 \sin C \\ &\quad \left. + a^2 \sin A + b^2 \sin B + c^2 \sin C\right\} - 2ABC \\ &= 2R^2\, \Sigma \sin^2 A\, \Sigma \sin A \\ &\quad + 4 \sin A \sin B \sin C\end{aligned}$$



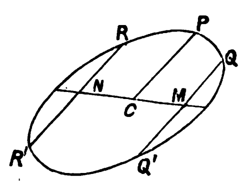
12050. (R. KNOWLES, B.A.)—CP is a radius of a conic; QQ′, RR′ are chords parallel to CP; prove that the line joining the poles of QQ′ and RR′ is parallel to the tangent at P.

Solution by J. F. HUDSON, M.A.; Prof. KOEHLER; *and others.*

Let M, N be the mid-points of QQ′, RR′. Then the poles of QQ′, RR′ lie on CM, CN produced.

Also, M, N, C are collinear, and MN is parallel to the tangent at P.

Hence, the join of poles of RR′, QQ′ is parallel to the tangent at P.



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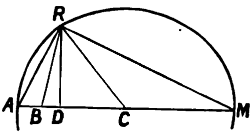
484. (J. H. SWALE.)—Let C be the centre, and B any point in the radius AC of the semicircle ARH; draw any line AR to the periphery, and join AR; then $AR^2 : BR^2 - BA^2 = AC : CB$; or $AR^2 : AB^2 - BR^2 = AC : CB$; according as B lies *between* or *beyond* A and C.

Solution by W. J. Greenstreet, M.A.; M. Brierley; *and others.*



Let RD be the perpendicular on the diameter AM; then we have

$$BR^2 - BA^2 = RD^2 + DB^2 - BA^2$$
$$= AD \,.\, DM + AD\,(DB - BA)$$
$$= 2BC \,.\, AD,$$

and $AR^2 = AM \,.\, AD = 2AC \,.\, AD$;

therefore $AR^2/(BR^2 - BA^2) = AC/BC$.

Similarly, when B lies beyond A and C.

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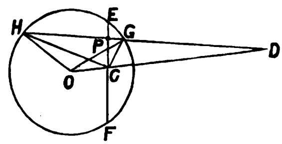
873. (Mortimer Collins.)—A point being taken within or without a circle, and in every chord passing through it a second point being taken, such that the chord produced is harmonically divided by these two points and the circumference, determine the locus of the latter points.

Solution by W. J. Greenstreet; Prof. Morel; *and others.*

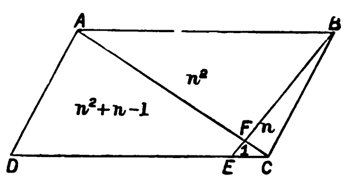


If DGH be any chord through the fixed point D, then, if EF is the polar of D, cutting HD in P, we know that P and D are harmonic conjugates to H and G. Hence the required locus is the polar of the given point with respect to the circle.

11911. (R. W. D. Christie.)—ABCD is a parallelogram. An nth part CE is cut off side CD. A diagonal AC cuts BE in F. Prove that the respective areas are respectively proportional to

trapezium ADEFA $= n^2 + n - 1$, triangles AFB, BFC, CFE $= n^2,\ n,\ 1$.

Solution by T. Savage; R. Chartres; *and others.*



Since the triangles FAB, FEC are similar, $AB = n \,.\, EC$, and

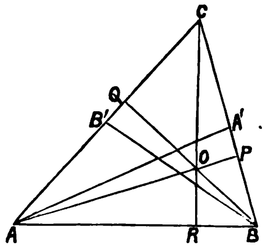
$$BF = n \,.\, FE;$$

$\therefore$ FEC : FBC : FAB : ADEFA

$$= 1 : n : n^2 : n^2 + n - 1.$$

12036. (Professor RINDI.)—Soient AA′, BB′ deux médianes du triangle ABC; démontrer que les cercles décrits sur AA′ et BB′ comme diamètres ont pour axe radical la hauteur de ABC qui correspond au sommet C.

Solution by J. M. STOOPS, B.A.; J. F. HUDSON; *and others.*



Let the perpendiculars AP, BQ, CR of the $\triangle ABC$ meet in O.

The circles on AA′, BB′ as diameters will pass through P, Q respectively.

Since AQPB is cyclic, therefore

$$CQ \cdot CA = CP \cdot CB;$$

therefore $CQ \cdot CB' = CP \cdot CA'$;

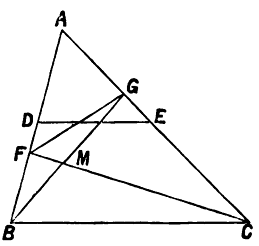
therefore C lies on the radical axis.

Also $\quad AO \cdot OP = BO \cdot OQ$;

therefore O lies on the radical axis; therefore COR is the radical axis.

12001. (Professor BERNÈS.)—Si, entre les côtés AB, AC d'un triangle ABC, on trace DE parallèle à BC, et FG antiparallèle relativement à l'angle A, l'axe radical des circonférences BEG, CDF est indépendant de la position de DE; il coïncide avec la droite qui joint A à la rencontre des droites BG, CF.

Solution by Prof. A. DROZ-FARNY; H. W. CURJEL, B.A.; *and others.*



Soit M le point d'intersection de FC et BG; le quadrilatère BFGC étant inscriptible, on a $MF \cdot MC = MB \cdot MG$; M est un point d'égale puissance par rapport aux deux circonférences considerées; il appartient donc à leur axe radical. On a de même

$$AF \cdot AB = AG \cdot AC.$$

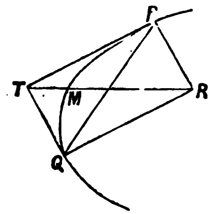
Or, comme $AB : AD = AC : AE$,

on a aussi $AF \cdot AD = AG \cdot AE$,

donc A appartient aussi à l'axe radical, qui est ainsi la droite AM.

11977. (R. TUCKER, M.A.)—Find (1) the locus of a point R, so that the normals therefrom to a parabola are mutually at right angles; and hence (2) derive a property of orthogonal tangents to a semi-cubical parabola.

Solution by H. W. CURJEL, B.A.; Prof. CHAKRIVARTI; *and others.*



1. Let the normals at P, Q to the parabola $y^2 = 4a(x-a)$ cut at right angles in R; then the tangents at P, Q cut at right angles at T (say); hence T is on the directrix; also TR, PQ bisect one another; therefore TR is a diameter and TR = 4TM, where TR cuts the parabola in M; therefore locus of R is

$$y^2 = 4a\left(\tfrac{1}{4}x - a\right), \quad \text{or} \quad y^2 = ax - 4a^2.$$

2. Hence we see that the locus of intersections of orthogonal tangents to a semi-cubical parabola (evolute of a parabola) is a parabola.

[*Otherwise*:—The PROPOSER remarks that the normal equation is $am^3 + (2a-x)m - y = 0$; whence, if m_1, m_2, m_3 are the points on the parabola, $\Sigma m = 0$, $\Sigma(m_1m_2) = (2a-x)/a$, $m_1m_2m_3 = y/a$, $m_1m_2 = -1$; whence $y = -am_3$, $x - 3a = am_3^2$; therefore the locus of R is $y^2 = a(x-3a)$.]

11858. (Professor VAUTRÉ.)—Dans un déterminant de Vandermonde $|1, a, a^2 \dots a^{n-1}|$ on remplace les éléments de la dernière colonne par $a^{n+p-1}, b^{n+p-1}, \dots l^{n+p-1}$. Démontrer que le déterminant correspondant est égal au déterminant de Vandermonde multiplié par la somme des combinaisons p à p, avec répétition des lettres $a, b, \dots l$.

Solution by Profs. CURTIS, CHAKRIVARTI, *and others.*

The Vandermonde determinant vanishes when $a = b$, &c., and therefore is $= -(a-b)(a-c)\dots(b-c)$. Now

$$\begin{vmatrix} 1 & a & a^2 & \dots\dots & a^{n-2} & a^{n+p-1} \\ 1 & b & b^2 & \dots\dots & b^{n-2} & b^{n+p-1} \\ \dots & \dots & \dots & & \dots & \dots \end{vmatrix}$$

also vanishes when $a = b$, &c.; and therefore

$$= \Delta_n (x \,.\, \Sigma a^p + y \,.\, \Sigma a^{p-1}.\, b + z \,.\, \Sigma a^{p-2}.\, bc + \dots),$$

where Δ_n = the Vandermonde. That $x = y = z = \dots 1$, may be proved as follows. As every term of the given determinant is different from every other term, and the same is true of Δ_n, therefore no term can contain a numerical factor, and therefore x, y, z, &c., must each be ± 1.

Again, $$\begin{vmatrix} 1 & a & a^2 & \dots\dots & a^{n+p-1} \\ 1 & b & b^2 & \dots\dots & b^{n+p-1} \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = \Sigma bc \dots \begin{vmatrix} 1 & b & \dots\dots & b^{n+p-2} \\ 1 & c & \dots\dots & c^{n+p-2} \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \Sigma bc \dots \Delta_{n-1} (x \,.\, \Sigma b^p + y \,.\, \Sigma b^{p-1} c + \dots),$$

$x, y, z \dots$ remaining the same as before (as may be seen by putting $a = 0$);

therefore the coefficients $x, y, z\ldots$ if found in the expansion when $n = k$, are also found when $n = k+1$. But, if $n = 2$,

$$\begin{vmatrix} 1 & a^{p+1} \\ 1 & b^{p+1} \end{vmatrix} = -(a-b)(a^p + a^{p-1}b + a^{p-2}b^2 + \ldots);$$

therefore $$x = y = z = \ldots = 1.$$

11955. (R. Knowles, B.A.)—Tangents TP, TQ are drawn to meet a parabola, vertex A, in P and Q; if O be the orthocentre of the triangle TPQ, and θ_1, θ_2 the inclinations of AO and PQ to the axis, prove that

$$\tan(\pi - \theta_1) = 2\cot\theta_2.$$

Solution by W. J. Dobbs, B.A.; Professor Bhattacharya; *and others.*

Let the coordinates of P be $(am^2, 2am)$, and the coordinates of $Q(an^2, 2an)$, $y^2 = 4ax$ being equation to parabola. The equation to the tangent at Q is $ny = x + an^2$; therefore the equation to the perpendicular from P on TQ is

$$nx + y = am\,(mn+2) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Similarly, the equation to the perpendicular from Q on TP is

$$mx + y = an\,(mn+2) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The equation to the line joining the origin to the point of intersection of (1) and (2) is $n\,(nx+y) = m\,(mx+y)$; $\therefore \tan(\pi - \theta_1) = m+n$.

The equation to PQ is $$\frac{x - am^2}{m^2 - n^2} = \frac{y - 2am}{2\,(m-n)}.$$

therefore $\cot\theta_2 = \frac{1}{2}(m+n)$, therefore $\tan(\pi - \theta_1) = 2\cot\theta_2$.

11691. (Professor Zerr.)—A volume V of gas at M mm. pressure and t° C. is saturated with aqueous vapour. Prove that its volume at the same pressure and T° C. is

$$\{V\,(H-f)(1 + \cdot 00367T^\circ)^2\} \,/\, \{(H-F)(1 + \cdot 00367t^\circ)^2\},$$

where f and F are the tensions of aqueous vapour in mm. of mercury at t° and T°; and find the mass (in grams) of water deposited by 1 litre of air saturated with aqueous vapour at 760 mm. and 30° C., if the temperature fall to 0°.

Solution by H. J. Woodall, A.R.C.S.; *the* Proposer; *and others.*

Using the well-known laws that volume varies inversely as pressure and directly as the absolute temperature, we find that

$$V' = \{V\,(H-f)(1+\alpha T)\} \,/\, \{(H-F)(1+\alpha t)\}, \text{ where } \alpha = \cdot 00367.$$

Balfour Stewart (*Heat*) gives the following formula for the weight of

the aqueous vapour present in the air at partial pressure P and temperature t (760 being normal pressure),

$$W = 0\cdot6235 \times 1\cdot293187P \,/\, \{760 \times (1+\alpha t)\}$$

$$\text{required precipitated weight} = \frac{0\cdot6235 \times 1\cdot293187}{760}\left\{\frac{31\cdot548}{1+30\alpha} - \frac{4\cdot600}{1}\right\}$$

$$= \cdot002526 \text{ grammes.}$$

3736. (Rev. T. Mitcheson, B.A.)—Find the area of the portion common to the two circles represented by the equation

$$x^4+y^4-16(x^3+xy^2)-32(y^3+x^2y)+2x^2(y^2+30)+16y(11x+12y)+44=0.$$

Solution by H. J. Woodall, A.R.C.S.

The equation resolves into

$$(x^2+y^2-2Ax-2By+C)(x^2+y^2-2A'x-2B'y+C')=0,$$

where $A, A' = 4 \pm \sqrt{5}$, $B, B' = 8 \pm 2\sqrt{5}$, $C, C' = 8 \pm 2\sqrt{5}$;

the distance between centres is 10, and the radii are $r, r' = (97 \pm 38\sqrt{5})^{\frac{1}{2}}$; thus the circles do not cut (since $10+r'$ is less than r).

If the circles cut, we could find the required area thus, the radii and distance being r, r_1, a.

If common chord subtended angles 2θ, 2ϕ at the centres, then

$$\cos\theta = (r^2+a^2-r_1^2)\,/\,2ar, \text{ \&c.,}$$

Required area = sum of areas of sectors—area of lozenge bounded by the radii $= \theta r^2 + \phi r_1^2 - ar\sin\theta$.

9722. (W. S. McCay, M.A.)—When all the angles and the perimeter to a convex polygon of any number of sides are given, prove that the area of the polygon is a maximum when it is circumscribed to a circle.

Solution by Professor Ramaswami Aiyar.

The angles alone being given, we have to show that (perimeter)^2/area is a minimum when the polygon circumscribes a circle.

Lemma.—From one vertex of a given figure F of perimeter p and area A, cut off a $\triangle$ by a line drawn in a fixed direction, leaving a polygon F′ of perimeter $p' = (p-x)$ and area $A' = A - \lambda x^2$. (Since $\Delta = \lambda x^2$, it follows that λ is a constant.) Now the minimum value of p'^2/A' = that of

$$(p-x)^2 \,/\, A - \lambda x^2 = p^2/A - 1/\lambda \;\ldots\ldots\ldots\ldots\ldots\ldots (\alpha).$$

From this we see that if F′ be a figure with given angles having $\frac{\text{perimeter}^2}{\text{area}}$ a minimum, then the figure F, formed by omitting one of the sides, has the same property $\ldots\ldots\ldots\ldots\ldots\ldots (\beta)$.

Also, if F and F′ are both circumscribed to a circle (of radius R suppose), the relation (α) is satisfied; for then

$$2A = p \times R, \quad 2A' = p' \times R, \quad \text{and} \quad 2\Delta = x \times R;$$

and therefore $\dfrac{p^2}{A} - \dfrac{1}{\lambda} = \dfrac{p^2}{A} - \dfrac{x^2}{\lambda x^2} = \dfrac{p^2}{A} - \dfrac{x^2}{\Delta} = \dfrac{p-x}{\frac{1}{2}R} = \dfrac{p'}{\frac{1}{2}R} = \dfrac{p'^2}{A'}$.

Hence, if F circumscribes a circle, so must F′ ....(γ).

From (β) and (γ) the theorem follows readily.

11899. (Professor MADHAVARAO.)—Eliminate x, y, z from the equations $(y-a)(z-a) = ea^2+fa+g$, $(z-b)(x-b) = eb^2+fb+g$,

$(x-c)(y-c) = ec^2+fc+g$, $(x-b)(y-c)(z-a) = (x-c)(y-a)(z-b)$,

and show that the result is $(b-c)(c-a)(a-b)(e-1) = 0$.

Solution by H. W. CURJEL, B.A.; H. W. GORDON; *and others.*

The fourth equation may be written

$$(b-a)\,xy + (c-b)\,yz + (a-c)\,xz + xa\,(c-b) + yb\,(a-c) + zc\,(b-a) = 0.$$

Substituting for yz, zx, xy from the first three equations, we get

$$\Sigma\left[(b-a)\left\{c(y+x)+(e-1)c^2+fc+g\right\}\right] + xa(c-b) + yb(a-c) + zc(b-a) = 0;$$

therefore $\qquad \Sigma\left\{(b-a)\,c^2\right\}(e-1) = 0,$

or $\qquad (b-c)(c-a)(a-b)(e-1) = 0.$

9703. (Professor MADHAVARAO, M.A.)—Show that the whole area of the first negative pedal of the ellipse (semi-axes a, b), with respect to its centre as origin, is $\qquad \pi\left\{\dfrac{(a^2+b^2)^2}{8ab} + \frac{1}{2}ab\right\}.$

Solution by Profs. ZERR, BHATTACHARYA, *and others.*

The first negative pedal of an ellipse with respect to its centre is the envelope of one side of a right-angled triangle having its right angle always on the circumference of the ellipse, and the other side passing through the centre. The equation of the line which envelopes the first negative pedal is

$$x\cos\theta + y\sin\theta = ab \,/\, \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)} \quad \text{...............(1).}$$

Differentiating (1) with respect to θ, we get

$$-x\sin\theta + y\cos\theta = -ab\,(a^2-b^2)\sin\theta\cos\theta \,/\, (a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{3}{2}} \text{...(2);}$$

squaring (1) and (2), and adding, we get

$$x^2+y^2 = r^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta} + \frac{a^2b^2\,(a^2-b^2)\sin^2\theta\cos^2\theta}{(a^2\sin^2\theta + b^2\cos^2\theta)^3},$$

where θ is the angle the perpendicular from the origin to the tangent makes with axis of x; therefore

$$A = \text{area} = 2ab \int_0^{\frac{1}{2}\pi} d\phi + \frac{2(a^2-b^2)^2}{ab} \int_0^{\frac{1}{2}\pi} \sin^2\phi \cos^2\phi \, d\phi,$$

where $\tan\phi = a/b \tan\theta$;

therefore $$A = \pi ab + \frac{\pi (a^2-b^2)^2}{8ab} = \pi \left\{ \frac{(a^2+b^2)^2}{8ab} + \tfrac{1}{2}ab \right\}.$$

11990. (Professor ORCHARD, M.A.)—Two conical vessels, of heights 12 inches and 18 inches, are filled with mercury up to heights 8 inches and 15 inches respectively, and their vertices are then fastened to the ends of a string which passes over a fixed pulley; find the total normal pressures on the bases during the motion.

Solution by H. W. CURJEL, B.A.; Professor SARKAR; *and others.*

Let A, B be the areas of the bases of the latter and former; then the latter moves downwards with an acceleration

$$\frac{645A-416B}{645A+416B} g;$$

hence, if m is the mass of a cubic inch of mercury, the pressures on the bases of the latter and former are

$$15\left(1-\frac{645A-416B}{645A+416B}\right) mg\,A, \qquad 8\left(1+\frac{645A-416B}{645A+416B}\right) mg\,B;$$

that is, $$\frac{12480\,mg\,AB}{645A+416B} \quad\text{and}\quad \frac{10320\,mg\,AB}{645A+416B}.$$

12077. (W. J. GREENSTREET, M.A.)—Prove that the series, general term $(n \log n \log\log n)^{-1}$, is divergent.

Solution by GERTRUDE POOLE, B.A.; Professor PAROLING; *and others.*

The series whose general term is $(n \log n \log\log n)^{-1}$ will be divergent if the series whose general term is $a^n \times (a^n \log a^n \log\log a^n)^{-1}$ is divergent, *i.e.*, if the series whose general term is $(\log a^n \log\log a^n)^{-1}$ is divergent. In this series

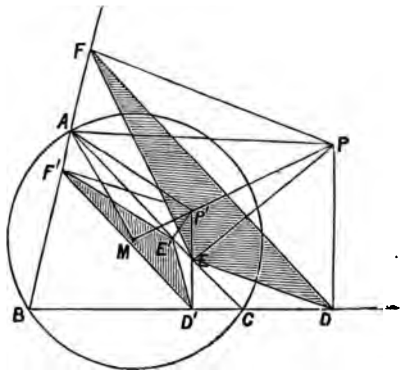
$$\text{Lim.}_{n=\infty} \frac{n^{\text{th}}\text{ term}}{(n-1)^{\text{th}}\text{ term}} = \text{L}_{n=\infty} \frac{(n-1)\log a + \log\log(n-1) + \log\log\log a}{n\log a + \log\log n + \log\log\log a}$$

$$= \text{L}_{n=\infty} \frac{\log a + 1/[(n-1)\log(n-1)]}{\log a + 1/(n\log n)} \text{ (by differentiating)} = 1.$$

Hence this series is divergent.

12113. (J. Griffiths, M.A.)—Prove, *geometrically*, that the pedal triangles with respect to a given triangle ABC of a pair of points inverse to each other with regard to the circumcircle ABC are similar. [This theorem is of importance in the geometry of the triangle and circle.]

Solution by Profs. Schoute, Mukhopadhyay, *and others.*



In the circles described on the diameters AP, AP′, the chords EF, E′F′ subtend the same angle A. This gives the relation

$$EF : E'F' = AP : AP'.$$

Moreover, the relation

$$MP \cdot MP' = MA^2$$

proves that the triangles AMP, P′MA are similar, and affords the relation

$$AP : AP' = PM : AM.$$

So we find

$$EF : E'F' = FD : F'D'$$
$$= DE : D'E' = PM : R,$$

where R stands for the circumradius, &c.

[Prof. Schoute states that he gave this proof many years ago in his paper, "Over een nauwer verband tusschen hoek en cirkel van Brocard," in the *Verslagen en mededeelingen* of the Academy of Amsterdam, Vol. iii., p. 59.

Mr. Griffiths remarks that he was not aware that the result had been noticed by Prof. Schoute. It had been suggested to the Proposer by the use of isogonal coordinates, and may be briefly expressed as follows, viz., $x + x' = 2\cos A$, $y + y' = 2\cos B$, $z + z' = 2\cos C$,

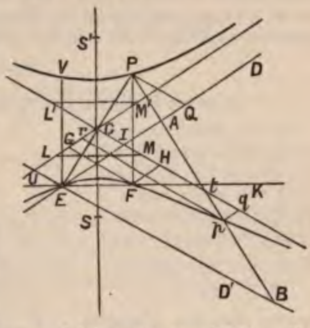
where $$x = \frac{\sin(D+A)}{\sin D}, \quad y = \frac{\sin(E+B)}{\sin E}, \quad z = \frac{\sin(F+C)}{\sin F};$$

$$x' = \frac{\sin(D-A)}{\sin D}, \quad y' = \frac{\sin(E-B)}{\sin E}, \quad z' = \frac{\sin(F-C)}{\sin F},$$

and the points P, P′ are collinear with the point (2 cos A, 2 cos B, 2 cos C). They are, in fact, inverse to each other with respect to the circumcircle ABC. Also D, E, F denotė the angles of the equiangular pedal triangles DEF, D′E′F′.]

12063. (Professor Cuny.)—Une droite mobile tourne autour d'un point fixe P; elle rencontre deux droites fixes D, D′ respectivement en A, B; trouver le lieu du milieu du segment AB. Ce lieu peut-il se réduire à des droites?

Solution by D. BIDDLE; H. W. CURJEL, B.A.; *and others.*



Let E be the point in which D and D′ meet. Join PE, and bisect it in C; also bisect the angle DED′ by EK, to which draw PF perpendicular. Complete the rectangle PFEV; we then have four points which fulfil the conditions required as mid-points. Moreover, if we draw, through C, lines parallel to D, D′, it can be shown that these are asymptotes to two hyperbolic curves, which are the loci required. A proof in regard to one, that passing through E, F, will suffice. Let p be the mid-point of AB. Draw pq, pr parallel to the supposed asymptotes, and PQ parallel to D′, meeting D in Q. Then $pq = \frac{1}{2}$AQ, and $tq = \frac{1}{2}$PQ = CI, whilst C$t = \frac{1}{2}$EB. Let AQ = one-mth of QE; then pq = one-mth of CU, EB = $(m-1)$ PQ, C$t = (m-1)$ CI, and C$q = m\,.$ CI. Wherefore $pq\,.\,pr$ = CU . CI, a constant. But this is a well-known characteristic of hyperbolic curves, in which $pq\,.\,pr = \frac{1}{4}\text{CS}^2$.

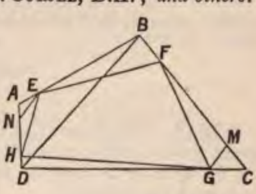
[*Otherwise*:—If we take D, D′ as axes of x and y, and put a, b for coordinates of P, the equation to PAB will be $y-b = m(x-a)$; hence the coordinates of A and B will be $x = 0$, $y-b = -ma$, and $y = 0$, $x-a = -m^{-1}b$; thus the locus of the mid-point of AB is the hyperbola $(2y-b)(2x-a) = ab$, or $2xy-bx-ay = 0$. This reduces to $2y = b$ when $a = 0$, and to $2x = a$ when $b = 0$; also, when D, D′ are parallel, the locus is evidently a straight line parallel to them and equidistant from them. The locus can evidently be rectilinear in no other case.]

12067. (EDITOR.)—If, in the sides AB, BC, CD, DA of a quadrilateral, points E, F, G, H be taken such that

$$\text{AE : EB = BF : FC = CG : GD = DH : DA},$$

prove that $$\triangle\text{AEH} + \triangle\text{CFG} = \triangle\text{BEF} + \triangle\text{DGH}.$$

Solution by GERTRUDE POOLE, B.A.; H. W. CURJEL, B.A.; *and others.*



Let AE : EB = $x : y$, where $x+y = 1$;

then $\triangle\text{AEH} = xy\triangle\text{ADB}$,

and $\triangle\text{CFG} = xy\triangle\text{BCD}$;

$\therefore\ \triangle\text{EHA} + \triangle\text{CFG} = xy\,(\text{ABCD})$.

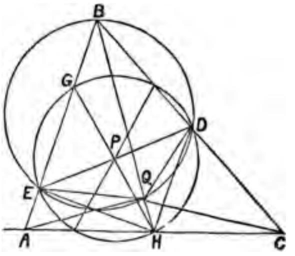
Similarly,

$\triangle\text{BEF} + \triangle\text{DGH} = xy\,(\text{ABCD})$;

$\therefore\ \triangle\text{AEH} + \triangle\text{CFG} = \triangle\text{BEF} + \triangle\text{GHD}$.

11593. (Professor CASEY, F.R.S.)—If a line through P, the symmedian point of the triangle ABC, meet the sides BC, BA in the points D, E, so that DP = PE, prove, if Q be the second point of intersection of the circumcircle of the triangle DBE and the circle described on AB as diameter, that the points B, C, Q and one of the Brocard points of ABC are concyclic. [Professor CASEY believes that the point Q possesses many properties in connexion with the triangle ABC.]

Solution by H. W. CURJEL, B.A.; Prof. CHAKRIVARTI; *and others.*



Let the cosine circle cut AB again in G and AC in H; G, P, H being collinear. And let Q be the intersection of BH with circle BED.

Then $\angle DQB = \angle DEB = \angle C$;

$\therefore$ D, Q, H, C are concyclic.

And $\angle HQD = \angle B + \angle A$,

but $\angle EQD = \angle A + \angle C$;

$\therefore$ $\angle EQH = \angle B + \angle C$;

$\therefore$ E, Q, H, A are concyclic;

$\therefore$ $\angle AQE = \angle AHE$

= complement of A.

But $\angle EQB = \angle EDB = \angle A$; $\therefore$ $\angle AQB$ is a right angle;

$\therefore$ Q is on the circle on AB as diameter.

Again, $\angle DQC = \angle DHC = \angle A$, and $\angle DQB = \angle C$;

$\therefore$ $\angle CQB = \angle A + \angle C$;

hence the circle BQC passes through one of the Brocard points.

11951. (J. YOUNG, M.A.)—Prove that the chance that the dealer and his partner at whist hold the four honours in trumps is greater than their chance of holding the four honours in any given suit (*e.g.*, hearts) in the ratio 5 : 4.

Solution by H. W. CURJEL, B.A.; Prof. KRISHNAMACHARRY; *and others.*

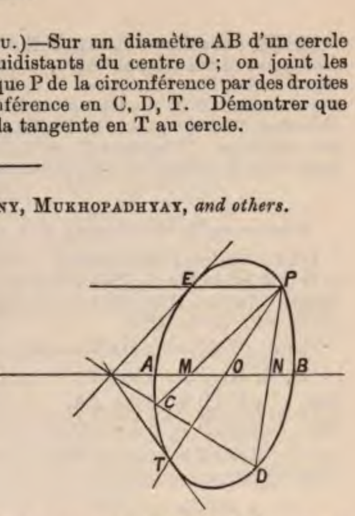
Chance that the dealer and his partner hold four honours in trumps : chance that they hold four honours in any given suit

$$= \frac{4}{13} \times \frac{25}{51} \times \frac{24}{50} \times \frac{23}{49} + \frac{9}{13} \times \frac{25}{51} \times \frac{24}{50} \times \frac{23}{49} \times \frac{22}{48} : \frac{1}{2} \times \frac{25}{51} \times \frac{24}{50} \times \frac{23}{49}$$

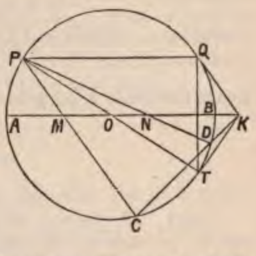
$$= \frac{4}{13} + \frac{9}{13} \times \frac{22}{48} : \frac{1}{2} = 5 : 4.$$

12064. (Professor FLEURANCEAU.)—Sur un diamètre AB d'un cercle O, on prend deux points M, N équidistants du centre O; on joint les points M, N, O à un point quelconque P de la circonférence par des droites qui rencontrent cette même circonférence en C, D, T. Démontrer que les droites AB, CD se coupent sur la tangente en T au cercle.

Solution b Profs. DROZ-FARNY, MUKHOPADHYAY, *and others.*



Prenons d'une manière plus générale une conique à centre O. Les couples de points M, N; A, B, &c., isotomiques par rapport au point O, appartiennent à une involution dont les points doubles sont O et le point infini sur AB. D'après un théorème bien connu, les rayons PM, PN; PA et PB, &c., coupent donc une conique passant par le sommet du faisceau suivant des cordes CD, AB, ... qui toutes passent par un même point, le point de coupe des tangentes en T et E où les rayons doubles PO et PE parallèle à AB rencontrent la conique.

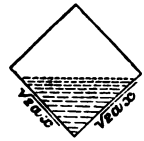


[*Otherwise*:—Draw PQ parallel to AB, cutting the circle in Q. PQT is a right angle; therefore the pole of QT lies on AB.

Since PO, PQ cut MN in its middle point and the point at infinity they are conjugate with respect to PM and PN; therefore the points (CD, TQ) are harmonic; therefore the tangents at Q, T both cut CD in the harmonic conjugate K of H, the point of intersection of TQ and CD; but K is the pole of QT, and therefore lies on AB; therefore CD and the tangent at T meet AB in the same point. The proof applies with very slight modifications to any central conic section.]

11967. (Professor HUDSON, M.A.)—A closed cubical vessel, whose edges (each $= a$) are vertical and horizontal, is filled to a depth x with water; it is then turned through 45° about a horizontal edge. Find the ratio of the whole pressure upon the faces of the cube in the former case to that in the latter. If $x = \frac{1}{4}a$, show that the ratio is $9 : 6\sqrt{2}+2$; and, if $x = \frac{3}{4}a$, that it is $45 : 78\sqrt{2}-70$.

Solution by C. MORGAN, M.A.; Prof. CHAKRIVARTI; *and others.*



Case 1.—When $x < \frac{1}{2}a$: Whole pressure

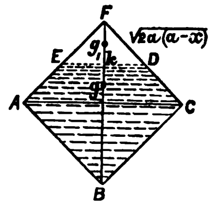
$$\propto a^2x + 2ax^2 \propto ax(a+2x).$$

When turned through angle of 45°: Whole pressure

$$\propto 2 \times \sqrt{2ax} \,.\, a \times \tfrac{1}{2}\sqrt{2ax} \times 1/\sqrt{2} + 2ax \times \tfrac{1}{3}\sqrt{ax},$$

$$\propto ax\left\{a\sqrt{2} + \tfrac{2}{3}\sqrt{ax}\right\};$$

therefore ratio $= 9/(6\sqrt{2}+2)$, where $x = \frac{1}{4}a$.



Case 2.—When $x > \frac{1}{2}a$: Whole pressure

$$\propto ax(a+2x) \propto \tfrac{15}{8}a^3, \quad x = \tfrac{3}{4}a.$$

When turned through an angle of 45°,

$$FE = FD = \sqrt{2a(a-x)}.$$

Whole pressure, if $x = \frac{3}{4}a$, $\propto$ 2 pressures on

$$(AB + AE + ABC + 2ACDE) = \frac{78\sqrt{2}-70}{24}a^3.$$

Therefore $\qquad$ ratio $= \dfrac{78\sqrt{2}-70}{24} \times \dfrac{8}{15} = \dfrac{78\sqrt{2}-70}{45}.$

Let g, g_1 be the centres of gravity of ACDE, FED,

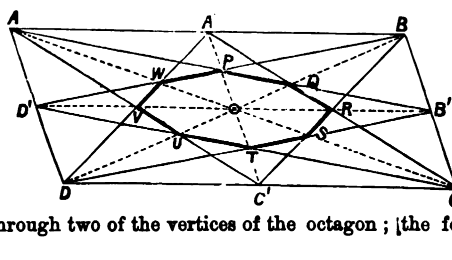
$$\Delta FED \times Fg_1 + (\Delta FAC - \Delta FED)\,Fg = \Delta FAC \,.\, \tfrac{2}{3}a/\sqrt{2};$$

therefore $\quad kg = Fg - \dfrac{\sqrt{2a^2-2ax}}{\sqrt{2}} = \dfrac{2\sqrt{2}}{3}a - \tfrac{1}{3}a - \tfrac{1}{2}a,$ if $x = \frac{3}{4}a$.

11973. (Professor VUIBERT.)—La surface de l'octogone formé par les huit droites qui joignent les sommets d'un parallélogramme aux milieux des côtés opposés est équivalente au sixième de l'aire du parallélogramme.

Solution by H. J. WOODALL, A.R.C.S.; Prof. DROZ-FARNY; *and others.*

If PQRSTUVW be the octagon, and A′B′C′D′ the mid-points of the sides of the parallelogram; join AC, BD, A′C′, B′D′; these lines will



each pass through two of the vertices of the octagon; the four will also meet in O.

We can easily prove that

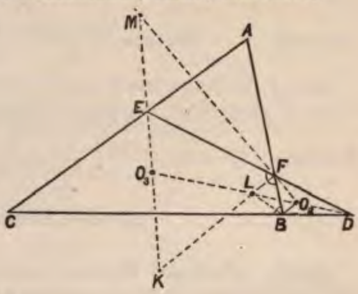
$$\text{OP} : \text{OA}' = \text{OR} : \text{OB}' = \text{OT} : \text{OC}' = \text{OV} : \text{OD}' = 1 : 2,$$

and $\quad \text{OW} : \text{OA} = \text{OQ} : \text{OB} = \text{OS} : \text{OC} = \text{OV} : \text{OD} = 1 : 3;$

therefore, dividing the octagon and the parallelogram into triangles by the lines AO, A′O, BO, B′O, CO, C′O, DO, D′O, we get eight pairs of triangles which are in the ratio $\text{OP} \times \text{OQ} : \text{OA}' \times \text{OB} = 1 : 6$; whence the theorem.

12103. (R. Tucker, M.A.)—A transversal DFE cuts the sides of the triangle ABC, viz., AC in E, AB in F, and CB produced in D. L, K are the escribed centres of BDF (to side BF) and AFE (to side FE); prove FLK a straight line. Again, O_2, O_3 are the in-centres of BDF, CED, and M is the ex-centre (to side AE) of AFE; prove ΔO_2O_3M to be of constant form.

Solution by T. Savage; W. J. Greenstreet, M.A.; *and others.*



Centre L lies on bisector of angle BFE; and centre K lies on same line; hence F, L, K are collinear. Again,

$$\angle \text{FO}_2\text{L} = \tfrac{1}{2}(\angle \text{D} + \angle \text{BFD})$$
$$= \tfrac{1}{2} \text{ supplement of B}$$
$$= \text{constant};$$

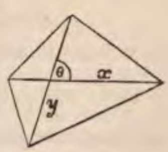
also $\quad \angle \text{MO}_3\text{O}_2$

$$= \angle \text{C} + \tfrac{1}{2}(\angle \text{D} + \angle \text{CED})$$
$$= \angle \text{C} + \tfrac{1}{2} \text{ supplement of C}$$
$$= \text{constant}.$$

Hence ΔO_2O_3M is of constant form.

12048. (R. F. Davis, M.A.)—Given the sides of a convex quadrilateral, prove that its area is greatest when the rectangle contained by its diagonals is greatest.

Solution by R. Chartres; Prof. Droz-Farny; *and others.*



(1) $xy \sin\theta = 2$ area; and (2) $xy \cos\theta =$ constant, the sides being given (Nixon's *Trigonometry*, p. 276); hence, when area = maximum, $\{(1)^2 + (2)^2\}^{\frac{1}{2}}$, or xy = maximum.

[*Otherwise* :—Parmi tous les quadrilatères que l'on peut construire avec quatre côtés donnés, on sait que le quadrilatère inscriptible est le plus grand. Mais on sait aussi que si D et D′ sont les diagonales d'un

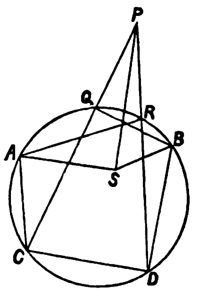
quadrilatère dont les côtés sont a, b, c, et d, et dont deux angles opposés sont θ et θ', on a (CASEY's *Seq. to Euc.*, Bk. VI., Quest. 102)

$$D^2D'^2 = a^2c^2 + b^2d^2 - 2abcd\cos(\theta + \theta').$$

Le maximum de DD' aura lieu pour $\theta + \theta' = 180°$, valeur qui correspond au maximum de la surface.]

12024. (Professor DUPORCQ.)—Soient A, B, C, et D quatre points fixes d'un cercle O, P un point quelconque du plan, Q et R les points où les droites PC et PD coupent le cercle O. Trouver le lieu du second point d'intersection des cercles PQB et PRA quand le point P varie d'une manière quelconque.

Solution by J. M. STOOPS, B.A.; GERTRUDE POOLE, B.A.; *and others.*



S is the second point of intersection of the circles ARP, BQP. Join S to A, B, P; then

$$\angle ASP = ARP = ACD,$$
$$\angle BSP = BQP = BDC,$$
$$\angle ASB = ACD + BDC,$$

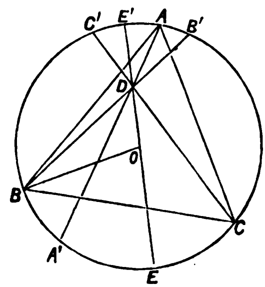
hence the locus of S is an arc described on AB as chord, containing an angle

$$= ACD + BDC.$$

12006. (Professor HAIN.)—Les droites joignant les sommets d'un triangle équilatéral ABC à un même point D recontrent la circonférence ABC aux points A', B', C'. Démontrer que

$$AD \cdot AA' + BD \cdot BB' + CD \cdot CC' = 2AB^2.$$

Solution by Profs. KRISHNAMACHARRY, DROZ-FARNY, *and others.*



Take the centre O, and draw the diameter E'DOE. Join OB. Now

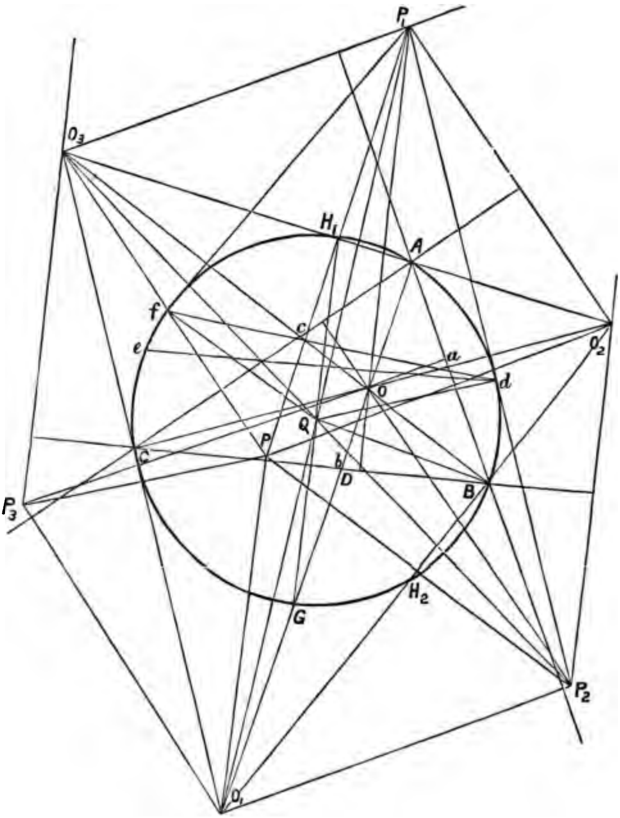
$$AD \cdot AA' + BD \cdot BB' + CD \cdot CC'$$
$$= AD^2 + BD^2 + CD^2 + 3AD \cdot DA'$$
$$= AG^2 + BG^2 + CG^2 + 3DG^2 + 3DE \cdot DE'$$
(where G is the centroid)
$$= AO^2 + BO^2 + CO^2 + 3DO^2 + 3(OE^2 - OD)$$
$$= 6BO^2 = 2AB^2,$$

as G and O coincide for the equilateral triangle.

12008. (Editor.)—Circles are drawn through the angles of a triangle, though the three escribed centres, and through the inscribed and each two of the escribed centres; show that the radical axes of these circles will meet the sides of the triangle at the points where they are cut by the bisectors of its angles.

Solution by M. Brierley; R. Knowles, B.A.; *and others.*

Let the figure be drawn as in the margin, O_1, O_2, O_3 being the centres of the escribed circles, Q, P, P_1, P_2, P_3 the centres of the other circles, and *a, b, c* the points where the bisectors meet the sides of the triangle.



These points being joined, draw *de* parallel to the base CB; then $\angle edacf = \angle bOD = CBOO_1 - BCOO_3$ (O being the centre of the inscribed

circle) = $P_1O_1P = O_1QP_1 = QP_1OD$. Therefore, since P_1OD is perpendicular to *de*, P_1Q is perpendicular to *dacf*, which is the radical axis of the circles P_1 and Q, because it is bisected by the line P_1Q, joining the centres P_1 and Q; and therefore $P_1d = P_1f = P_1O = P_1O_2 = P_1O_3$. Similarly can it be shown that the radical axes of the circles P_1, P_2, P_3 pass through points *a*, *b*, *c*.

The circles P_1, P_2, P_3 obviously cut the circle Q orthogonally.

11993. (Professor DE LONGCHAMPS.) — Sur deux droites Δ, Δ' on considère deux points fixes A, A′. Soient B, B′ deux points mobiles tels que le quadrilatère ABA′B′ soit inscriptible à un cercle. Par A, B on mène des parallèles aux bissectrices des angles des droites Δ, Δ'. La diagonale δ du rectangle ainsi formé, et celle du rectangle analogue construit avec A′B′, concourent en un point dont le lieu géométrique est une droite passant par le milieu de AA′.

Solution by Professors DROZ-FARNY, BHATTHACHARYA, *and others.*

Comme on le voit immédiatement δ est parallèle à la droite Δ' et δ' est parallèle à Δ. Le lieu géometrique des centres des circonférences étant la droite perpendiculaire à AA′ en son point milieu, les points milieux des segments AB et A′B′, projections sur Δ et Δ' des centres des circonférences, décriront deux ponctuelles projectives ayant les éléments à l'infini comme points correspondants. δ et δ' décrivent par conséquent deux faisceaux perspectifs. Le lieu cherché est donc une droite qui passe par le milieu de AA′; car, pour la circonférence AOA′, δ et δ' se coupent évidemment au point milieu considéré.

Lorsque AO = OA′, la ligne des centres coincide avec le lieu cherché et passe en outre par O.

12096. (Professor NEUBERG.)—On donne dans un même plan deux droites AB, CD et un point M. On construit le triangle CDM′ semblable à ABM, le triangle CDM″ semblable à ABM′, le triangle CDM‴ semblable à ABM″, &c. Démontrer que les points M, M′, M″, M‴, &c., appartiennent à une même spirale logarithmique.

Solution by Profs. SCHOUTE, DROZ-FARNY, *and others.*

Soit O le point double des figures semblables (AB) et (CD). Représentons par $\sigma e^{i\beta}$, $\tau e^{i\gamma}$ les vecteurs OC, OM par rapport à OA comme unité. Alors le vecteur du point $M^{(n+1)}$ est $\sigma^n \tau e^{i(n\beta+\gamma)}$. Donc les coordonnées polaires de ce point sont $\rho = \sigma^n\tau$, $\phi = n\beta + \gamma$. En éliminant n on obtient l'équation $\log \frac{\rho}{\tau} \Big/ \log\sigma = (\phi-\gamma)/\beta$, d'une spirale logarithmique.

12086. (A. E. Thomas, M.A.)—Show how to obtain integral solutions of the equations $x^3+y^3=z^2$, $x^3+y^3=z^3$, $x^3+y^3=z^4$.

Solution by A. Martin, LL.D.; Prof. Beyens; *and others.*

I will consider the last equation first. Let $az=x$ and $bz=y$; then we have $a^3z^3+b^3z^3=(a^3+b^3)z^3=z^4$; $z=a^3+b^3$;

$$x=a(a^3+b^3), \quad y=b(a^3+b^3).$$

$$x^3+y^3=a^3(a^3+b^3)^3+b^3(a^3+b^3)^3=(a^3+b^3)(a^3+b^3)^3=(a^3+b^3)^4.$$

These values of x and y also satisfy the first equation. The second equation, $x^3+y^3=z^3$, has been proved by Euler and others to be impossible in rational numbers.

4635. (Professor Hudson, M.A.)—Find (1) the envelope of the pedals of a given curve when the given point moves along another curve, and prove that it is the envelope of the pedals of the latter curve when the given point moves along the former; and (2) find this envelope if the curves are $y^2=4ax$, $x^2+y^2=a^2$.

Solution by H. J. Woodall, A.R.C.S.

The required curve is the locus of intersection of perpendicular tangents, whence its reciprocal character. In the given case, tangents to $y^2=4ax$, $x^2+y^2=a^2$, are $mx-y+a/m=0$, $x\cos\theta+y\sin\theta-a=0$; and these tangents are perpendicular if $m=\tan\theta$. Then the equations become $m^2x-my+a=0$, $(x+y\tan\theta)^2=a^2\sec^2\theta=a^2(1+\tan^2\theta)$;

therefore $$m^2(y^2-a^2)+2xym+(x^2-a^2)=0.$$

Eliminating m, we have

$$(x^3-ay^2-a^2x+a^3)^2+(2xya+yx^2-a^2y)(2x^2y+y^3-a^2y)=0,$$

or $$\begin{vmatrix} y^2-a^2, & 2xy, & x^2-a^2, & 0 \\ 0, & y^2-a^2, & 2xy, & x^2-a^2 \\ x & -y, & a, & 0 \\ 0, & x, & -y, & a \end{vmatrix}=0.$$

3858 & 3913. (R. Tucker, M.A.)—A blind man and his dog traverse the same curved path, the connecting string being kept stretched. Find (1) the curve which a stick carried by the dog at a constant inclination to the string always touches; and (2) when the two curves will be the same or similar; and (3) discuss the curves obtained by supposing the dog and man to move in confocal conics.

Solution by H. J. WOODALL, A.R.C.S.

1. Let $y = f(x)$ be the curve, (x_1, y_1), (x_2, y_2) two simultaneous points; then the eq ation to the joining line is

$$(X - x_1)(y_2 - y_1) = (Y - y_1)(x_2 - \quad \text{.....................(1)}.$$

If $(X - x_1) = m\,(Y - y_1)$ meets (1) at an angle $\alpha = \tan^{-1} k$, then

$$m = \left[(x_2 - x_1) + k\,\{f(x_2) - f(x_1)\}\right] / \left[\{f(x_2) - f(x_1)\} - k\,(x_2 - x_1)\right].$$

The envelope required is, therefore, envelope of

$$(X - x_1)\,\left[f(x_2) - f(x_1) - k\,(x_2 - x_1)\right]$$
$$= (Y - y_1)\,\left[(x_2 - x_1) + k\,\{f(x_2) - f(x_1)\}\right] \text{.........(2)},$$

with the condition that

$$(x_2 - x_1)^2 + \{f(x_2) - f(x_1)\}^2 = a^2 = (\text{length of string})^2 \text{.........(3)}.$$

2. To get envelope, we must eliminate x_2 between these equations, and get, say, $F(x_1) = 0$. Then eliminate x_1 between $F(x_1) = 0$ and $F'(x_1) = 0$.

3. Here (2) is

$$(X - a\cos\theta)\,\left[(b^2 + \lambda)^{\frac{1}{2}}\sin\phi - b\sin\theta - k\,\{(a^2 + \lambda)^{\frac{1}{2}}\cos\phi - a\cos\theta\}\right]$$
$$= (Y - b\sin\theta)\,\left[(a^2 + \lambda)^{\frac{1}{2}}\cos\phi - a\cos\theta + k\,\{(b^2 + \lambda)^{\frac{1}{2}}\sin\phi - b\sin\theta\}\right].$$

(3) is $\{(a^2 + \lambda)^{\frac{1}{2}}\cos\phi - a\cos\theta\}^2 + \{(b^2 + \lambda)^{\frac{1}{2}}\sin\phi - b\sin\theta\}]^2 = A^2.$

Eliminate ϕ, and get $F(\theta) = 0$; then eliminate θ between

$$F(\theta) = 0 \quad \text{and} \quad F'(\theta) = 0.$$

12013. (R. CHARTRES.)—Obtain a simple rule for finding approximately the number of years in which a sum of money will double itself at any ordinary per cent. compound interest.

Solution by H. J. WOODALL, A.R.C.S.; M. BRIERLEY; *and others.*

The formula is $2 = (1 + r)^x$, where r is rate in decimals; therefore

$$x = \log 2 / \log(1 + r) = \log 2 / (r - \tfrac{1}{2}r^2) \text{ approximately,}$$

where $\log 2 = \cdot 69315$; and x is easily calculable. [It is nearly $= 70/$(rate per cent.); thus the rule is to divide 70 by rate per cent.]

11903. (Professor CATALAN.)—Simplifier le polynome

$$(1 - x)(1 - x^2)(1 - x^3)\ldots(1 - x^n) + x\,(1 - x^2)(1 - x^3)\ldots(1 - x^n)$$
$$+ x^2\,(1 - x^3)(1 - x^4)\ldots(1 - x^n) + \ldots + x^n.$$

Solution by H. W. CURJEL, B.A.; Professor ZERR; *and others.*

The given expression

$$= (1-x^2)(1-x^3)\dots(1-x^n)+x^2(1-x^3)(1-x^4)\dots(1-x^n)+\dots+x^n$$
$$= (1-x^3)(1-x^4)\dots(1-x^n)+\dots+x^n = 1.$$

4786. (A. MARTIN, LL.D.)—Find the mean distance of a given point in the surface of a circle (1) from all points in its circumference, and (2) from all other points in its surface.

Solution by H. J. WOODALL, A.R.C.S.

1. Let given point be $(h, 0)$, then required distance is

$$\int_0^\pi \left\{(h-a\cos\theta)^2+a^2\sin^2\theta\right\}^{\frac{1}{2}} d\theta \Big/ \int_0^\pi d\theta = \frac{1}{\pi}\int_0^\pi (h^2+a^2-2ah\cos\theta)^{\frac{1}{2}} d\theta.$$

2. Distance $= \left\{\int_0^a \frac{1}{\pi}\int_0^\pi (h^2-2xh\cos\theta+x^2)\ \ d\theta\, x\, dx\right\} \Big/ \int_0^a x\, dx$

$$- \frac{2}{\pi a^2}\int_0^a\int_0^\pi (h^2-2hx\cos\theta+x^2)\ \ d\theta\, x\, dx.$$

7319. (R. RUSSELL, B.A.)—1. P, Q, R are three points on a conic S; P_1, Q_1, R_1 their corresponding points on a confocal S_1; if the tangents at Q and R intersect in P_1, then P, Q_1, R_1 are collinear.

2. Again, if we take any two confocals whose axes are (a, b) and (a_1, b_1), and denote by S the conic $\dfrac{x^2}{aa_1}+\dfrac{y^2}{bb_1}=1$, then the polar of any point on one confocal is the tangent at the corresponding point on the other.

3. If P, Q be any two points on one conic, P_1, Q_1 their corresponding points on the other: then, if their tangents at P, Q_1 intersect at right angles in R, so also do those at P_1, Q intersect in R_1, and the chord PQ_1 is the polar of R with respect to $S \equiv \dfrac{x^2}{aa_1}+\dfrac{y^2}{bb_1}-1=0.$

Solution by H. J. WOODALL, A.R.C.S.

1. Tangent at P is $bx\cos\theta+ay\sin\theta=ab$; this passes through Q_1, R_1, $(a_1\cos\phi,\ b_1\sin\phi)$, if $ba_1\cos\theta\cos\phi+ab_1\sin\theta\sin\phi=ab$, which is the condition that tangents at Q, R, $(a\cos\phi,\ b\sin\phi)$ should intersect at P_1, $(a_1\cos\theta,\ b_1\sin\theta)$. (Holds for any pair of conics.)

2. Polar of $(a\cos\theta,\ b\sin\theta)$ is $xa\cos\theta/aa_1+yb\sin\theta/bb_1=1$, *i.e.*, is the tangent at $(a_1\cos\theta,\ b_1\sin\theta)$. (Holds for any pair of conics.)

3. Tangent at P_1 is $x\cos\theta/a_1 + y\sin\theta/b_1 = 1$ ⎫ are perpendicular if
,, Q is $x\cos\phi/a + y\sin\phi/b = 1$ ⎭ $bb_1 + aa_1\tan\theta\tan\phi = 0$,
,, P is $x\cos\theta/a + y\sin\theta/b = 1$ ⎫
,, Q_1 is $x\cos\phi/a_1 + y\sin\phi/b_1 = 1$ ⎭ ,, ,,

Chord PQ_1 is
$$\frac{x - a\cos\theta}{a_1\cos\phi - a\cos\theta} = \frac{y - b\sin\theta}{b_1\sin\phi - b\sin\theta},$$
i.e., $x(b_1\sin\phi - b\sin\theta) + y(a\cos\theta - a_1\cos\phi) = ab_1\cos\theta\sin\phi - a_1b\sin\theta\cos\phi$,
this is polar of S with respect to point (x_1, y_1) given by
$$\frac{b_1\sin\phi - b\sin\theta}{x_1/aa_1} = \frac{a\cos\theta - a_1\cos\phi}{y_1/bb_1} = \frac{ab_1\cos\theta\sin\phi - a_1b\sin\theta\cos\phi}{1},$$
which give $x_1/aa_1 = $ &c., $y_1/bb_1 = $ &c., the point of intersection of tangents at P_1 and Q. (Holds for any pair of conics.)

12028. (Professor DROZ-FARNY.)—Par un point fixe de l'axe d'une parabole on mène une sécante variable qui coupe cette dernière aux points A et B. On construit les circonférences qui passent par le sommet de la parabole et lui sont tangentes en A et B. Chercher le lieu de leur second point d'intersection.

Solution by R. KNOWLES, M.A.; M. BRIERLEY; *and others.*

The equation to a circle passing through the vertex of the parabola $y^2 = 4ax$, and touching it at a point (x_1, y_1) is
$$x^2 + y^2 - (4a + 3x_1)\,x + y_1^3\,y/8a^2 = 0,$$
and similarly at the point B, (x_2, y_2); let h, k be the coordinates of the pole of AB, where h is constant; then, adding and subtracting the equations to the circles, we obtain
$$2\,(x^2+y^2) - 8ax - 3x\,(k^2 - 2ah)\,/\,a + 8k\,(k^2 - 3ah)\,y = 0 \quad\text{.........(1)},$$
$$3akx - (k^2 - ah)\,y = 0 \quad\text{..............................(2)};$$
substituting the values of k obtained from (2) in (1), we get
$$k = a\,(x^2 + y^2 - 4ax + 3hx)\,/\,hy,$$
and the locus is the quartic
$$a\,(x^2 + y^2 - 4ax)^2 + 3ahx\,(x^2 + y^2 - 4ax) - h^3y^2 = 0.$$

11780. (R. KNOWLES, B.A.)—PQ, PC are respectively the normal and curvature chords at the point P of a rectangular hyperbola; the circle on PQ as diameter cuts the hyperbola again in R; prove that (1) the figure PQRC is a parallelogram; (2) the diameter parallel to PQ bisects PC; (3) all chords parallel to PQ subtend a right angle at P.

Solution by Profs. Droz-Farny, Czuber, *and others.*

Un angle droit pivote autour de son sommet P. D'après le théorème de Frégier il intercepte sur la courbe une corde passant par un point fixe de la normale en P. Pour la position particulière des côtés de l'angle droit parallèle aux asymptotes, la droite infinie se confond avec la corde ; il en résulte que toutes les cordes sont parallèles à la normale au point P.

Par conséquent (3) toutes les circonférences décrites sur un système de cordes parallèles comme diamètres passent par 2 points fixes P et R de l'hyperbole dont les normales sont parallèles à la direction des cordes. P et R sont évidemment les extrémités d'un même diamètre de l'hyperbole. En considérant PQ comme une des cordes du système la circonférence sera tangente en P à l'hyperbole et coupera encore cette dernière au point diamétralement opposé R ; la corde RQ est donc perpendiculaire sur PR. La corde de courbure PC est perpendiculaire en P sur le diamètre PR car d'après un théorème connu, les cercles tangents en un point P de l'hyperbole équilatère coupent cette dernière suivant des cordes perpendiculaires au diamètre passant par le point P.

Les 2 cordes PC et RQ étant perpendiculaires aux points extrèmes d'un diamètre sont égales et parallèles. La figure PQRC est donc un parallélogramme dans lequel les 2 diagonales sont 2 diamètres de la courbe ; il en résulte immédiatement la propriété (2).

11771. (J. O'Byrne Croke, M.A.)—In a magic square composed of the simple factors of the expression $n(n^2-1^2)(n^2-2^2)\dots(n^2-r^2)$, arranged in $2r+1$ compartments, r being $\surd$(even), show that the sum of the quantities in the outer border compartments is equal to $4\left[\surd(2r+1)-1\right]n$.

Solution by Professors Zerr, Madhavarao, *and others.*

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

The factors are arranged as in the figure. When we take a square of nine compartments, then the upper and lower row each contain $\quad 15 = n\surd(2r+1)$.

Since $\quad n = 5$ and $2r+1 = 9$.

Then there still remains in the outside border 10, but

$$10 = 2\times 5 = 2\left[\surd(2r+1)-2\right]n.$$

If we take a square of 25 compartments $n = 13$, $2r+1 = 25$, then

$$n\surd(2r+1) = 65.$$

There still remains in the outer border $78 = 2\times 39 = 2\left[\surd(2r+1)-2\right]n$.

If we use a square of 49 compartments, $n = 25$, $2r+1 = 49$, then $n\surd(2r+1) = 175$. But there still remains in the outer border

$$250 = 2\times 125 = 2\left[\surd(2r+1)-2\right]n.$$

In each case the outer border contains

$$2 \times \sqrt{(2r+1)} \times n + 2\left[\sqrt{(2r+1)} - 2\right] n = 4\left[\sqrt{(2r+1)} - 1\right] n.$$

But, if it is true for these simple cases, it is true for 81, 121, ... $2r+1$ compartments.

[The PROPOSER remarks that if, in the square which is given as of an odd number of compartments, n be the quantity in the central compartment, then, filling up the other compartments with the remaining simple factors of the given expression, the sum of any two quantities in the outer border situated *diagonalwise* at equal distances from opposite angles is $= 2n$.]

11913. (W. J. DOBBS, B.A.)—If α, β, γ be real positive angles, such that $8 \sin\alpha . \sin\beta . \sin\gamma = 1$, $\alpha+\beta+\gamma = 90^\circ$, prove that $\alpha = \beta = \gamma = 30^\circ$

Solution by R. CHARTRES; Professors ZERR; *and others.*

If A, B, C be the angles of a triangle, then $8 \sin \frac{1}{2}A \sin \frac{1}{2}B \frac{1}{2} \sin C$ is less than unity, unless $A = B = C$ (NIXON'S *Trigonometry*). Hence

$$\alpha = \beta = \gamma = 30^\circ.$$

11786. (Professor SIRCOM.)—Prove that

$$p(p+n)^{n-1} + nrp(\ \ +n-1)^{n-2} + n\tfrac{1}{2}(n-1)\,rp\,(p+n-2)^{n-3}(r+2) + \dots$$
$$+ n\tfrac{1}{2}(n-1)\dots(n-s+1)/s\,.\,rp\,(p+n-s)^{n-s-1}(r+s)^{s-1} + \dots$$
$$\dots + nrp\,(r+n-1)^{n-2} + r(r+n)^{n-1} = (p+r)(p+\ \ +n)^{n-1};$$

where n is a positive integer.

Solution by H. J. WOODALL, A.R.C.S.

By ABEL'S theorem (§ 1573 of CARR'S *Synopsis*)

$$p(p+n+r)^{n-1} = p(p+n)^{n-1} + C(n-1, 1)\,rp\,(p+n-1)^{n-2}$$
$$+ C(n-1, 2)\,rp\,(r+2)(\ \ +n-2)^{n-3} + \&c.$$
$$+ C(n-1, s)\,rp\,(r+s)^{s-1}(p+n-s)^{n-s-1} + \&c.$$
$$+ C(n-1, n-2)\,rp\,(r+n-2)^{n-3}(p+2) + C(n-1, n-1)\,rp\,(r+n-1)^{n-2},$$
$$r(p+n+r)^{n-1} = r(r+n+p)^{n-1} = r(r+n)^{n-1} + C(n-1, 1)\,rp\,(r+n-1)^{n-2}$$
$$+ C(n-1, 2)\,pr\,(p+2)(r+n-2)^{n-3} + \&c.;$$

adding, we get the result.

11987. (Rev. Dr. KOLBE.)—Find a short method, analogous to that of Question 11868, which shall apply to fractions whose denominator ends in 1.

Solution by the PROPOSER.

The "artificial" division is performed in the same way by the number of tens as in Question 11868; only the last unit of the original dividend is regarded as $\cdot\dot{9}$, and at each step of division the quotient-figure is subtracted from 9 to make the new dividend. Thus (writing out the $\cdot\dot{9}$ for clearness) $\frac{4}{31}$ would be

$$\begin{array}{r} 3\,\underline{)\;3\cdot9999999999999999} \\ \cdot\dot{1}2903225806451\dot{6}1 \\ 0200001201110100 \end{array}$$

Reading it thus:—3 into 3, 1 and 0 over; $9-1=8$; 3 into 8, 2 and 2 over; $9-2=7$; 3 into 27, 9 and 0 over; &c.

For such numbers as 401, 3001, &c., the same process is followed, only the artificial division is carried one step back, as in Question 11868.

By this process many other fractions are simplified: *e.g.*, $\frac{1}{367}=\frac{3}{1101}$; therefore dividing artificially for 45 places by 11, we need after that divide only by 2, and the decimal circulates in the full round of 366 figures.

[Dr. KOLBE states that through our correspondent, Dr. MUIR, he has learned more of what has been done in circulating decimals, and found that several things which he himself thought were discoveries were already well known; but so far as Dr. KOLBE can find out, this method of *artificial* division is new.]

11860. (Professor CATALAN.)—Trouver les solutions entières des équations $x+y=u^2$, $x^2+y^2=r^2$.

Solution by A. MARTIN, LL.D.; Prof. AIYAR; *and others.*

Put $y=q/p\,.\,x$, then $\quad x+q/p\,.\,x=\square=u^2$ ....(3),

$$x^2+q^2/p^2\,.\,x^2=\square=r^2,\ =s^2x^2 \text{ say (putting } r=sx) \text{.........(4).}$$

From (3), $x=pu^2/(p+q)$; whence, dividing (4) by x^2, we get

$$(p^2+q^2)/p^2=s^2 \text{ ....(5),}$$

which is satisfied by $p=m^2-n^2$, $q=2mn$; for then $s=(m^2+n^2)/(m^2-n^2)$.

Let $m=2$, $n=1$; then $p=3$, $q=4$, and $s=\frac{5}{3}$, and we have $x=\frac{3}{7}u^2$, $y=\frac{4}{7}u^2$, where, for integers, u must be 7 or a multiple of 7. Take $u=7$, and we have $x=21$, $y=28$.

An infinite number of other integral values for x and y may be found by varying the values of m and n.

APPENDIX.

UNSOLVED QUESTIONS.

2792. (Professor Sylvester.)—If

$$r = \frac{(\theta^2 \pm A\theta + B)^i (\theta^2 \pm C\theta + D)^j}{(\theta^2 \pm E\theta + F)^{i+j}},$$

where i, j are any given positive or negative integers, and $\pm A \pm C \pm E = 0$, be the polar equation to a spiral, show that there are two, and only two, systems of values of A, B, C, D, E, F for which the curve will be rectifiable, and that in one of these systems $A = C = E = 0$.

2794. (R. Moon, M.A.)—If we have

$$0 = \frac{d^2z}{dy^2} - 2\alpha\frac{d^2z}{dx\,dy} + \alpha^2\frac{{}^2z}{dx^2} + \beta\frac{dz}{dy} + \gamma\frac{dz}{dx} + f(xyz),$$

where α, β, γ are functions of x and y only, show that the condition—requisite and sufficient—to be satisfied by the coefficients, in order that the equation may be integrable by Monge's method, is

$$0 = \frac{d\alpha}{dy} - \alpha\frac{d\alpha}{dx} + \beta\alpha + \gamma;$$

also that when the condition is satisfied the first integral given by Monge's method will be $0 = q - \alpha p + F(xyz)$, where the form of F is determined by the equation $0 = F'(y) - \alpha F'(x) - F \cdot F'(z) + \beta x - f(xyz)$.

2805. (G. O. Hanlon.)—In a given straight line find the point at which a given ellipse subtends the greatest possible angle.

2807. (Professor Crofton, F.R.S.)—Given any two convex boundaries of lengths L and L′, the latter inside the former; if Ω be the area of the inner, and if we denote the function $x - \sin x$ by u_x, prove that

$$\iint (u_{\theta+\phi} + u_{\theta+\psi} - u_\phi - u_\psi)\, dS + 2\iint \theta\, dS = LL' - 2\pi\Omega,$$

where θ is the angle between two tangents drawn to the curve L′ from any point P, and ϕ, ψ the angle between the same tangents, and two other tangents drawn from P to the curve L, the first integral extending over the whole space outside L, and the second over the space between L and L′.

2819. (Professor Crofton, F.R.S.)—Three straight lines are drawn at random across a given circle. Find the probability of their three intersections lying within the circle. [Several other problems relating to the positions of these lines will suggest themselves.]

2824. (Professor Sylvester.)—If

$$x + y + z = m, \quad \xi + \eta + \zeta = \mu,$$

$$\begin{aligned} i(y+z) + i(z+x) + k(x+y) &= xy + xz + yz - \xi\eta - \xi\zeta - \eta\zeta \\ &= i(\eta+\zeta) + j'(\zeta+\xi) + k'(\xi+\eta), \end{aligned}$$

$$iyz + izx + kxy = xyz - \xi\eta\zeta = i\eta\zeta + j'\zeta\xi + k'\xi\eta,$$

where $j' + k' = i + k$; prove that the system is of the eighth order (*i.e.*, admits of eight solutions), and that the values of x, y, z ; ξ, η, ζ depend on the solution of a biquadratic equation.

2848. (Professor Wolstenholme, Sc.D.) — Determine the condition which must hold in order that it may be possible to describe a conic round a given quadrangle the normals to which at the corners shall meet in a point; and, given three of the points, determine the locus of the fourth. [One solution is that the four points shall lie on a circle.]

2850. (M. Collins, B.A.)—If A, B, C, D, E be five points in space, and if a = vol. of pyramid BCDE, b = vol. of pyramid ACDE, &c.; then the condition that one of the five points should be within the pyramid formed by the other four, is

$$\tfrac{1}{2}S_1S_4 = S_2S_3 - S_5 + 4ABCDE, \quad \text{where} \quad S_n = a^n + b^n + c^n + d^n + e^n.$$

2859. (Professor Sylvester.)—Let $2i-1$ distinct logarithmic waves $y = c\log(x-\alpha)$, be given coaxal, but with their asymptotes disposable at will along the axis. Show that, with the aid of a new wave of the same form, the $2i$ asymptotes may be so interspaced (and that, too, in a variety of ways) as that the curve which represents the combined wave shall be rectifiable as a linear function of the abscissa and ordinates of the partial waves. Show also that *in general* the new partial wave introduced to complete the system admits of $2\dfrac{\Pi(2i-1)}{\Pi i\,\Pi(i-1)}$ distinct forms, and that to each of those forms correspond $\Pi i\,\Pi(i-1)$ modes of interspacing the asymptotes.

2861. (M. Collins, B.A.)—Subtract the sum of every two of the five numbers A, B, C, D, E from the sum of the remaining three of them, and we thus obtain ten different remainders (one of which is $A+B+C-D-E$); find the symmetric function which is the continued product of these ten remainders; and if $aA^8B^2 + bA^8BC + cA^7B^3 + dA^7B^2C + eA^7BCD + fA^6BCDE + gA^2B^2C^2D^2E^2 +$ &c. be a few terms of the said product, prove that $a=-3$, $b=6$, $c=8$, $d=0$, $e=-16$, and *especially* show how to find f and g.

2862. (Professor Crofton, F.R.S.)—If $l\rho + m\sigma + n\tau = 0$ represent the serpentine branch of a circular cubic, show that $l\rho^2 + m\sigma^2 + n\tau^2 = 0$ represents the tangent to the curve at the centre of the four concyclic foci.

2878. (S. Roberts, M.A.)—(1) Given a pencil of rays and a system of concentric circles, prove that if one set of intersections range on a straight line, the other intersections lie on two circular cubics, each having a double point at the origin of the pencil and the double focus at the common centre of the circle. (2) Hence determine, with reference to a system of parabolas having the same focus, the locus of the points the normals at which intersect in a fixed point.

2890. (Professor Sylvester.)—Let f, g be any two given incommensurable quantities, and U, V two rational integral functions, each of degree n in x^2, so taken as to cause the arc of the curve $y = f\log U + g\log V$ to become a linear function of x and of the logarithms of the factors of U and V. Prove that the number of curves fulfilling this condition (change by translation parallel to the axis of y being of course disregarded) is

$$1 \text{ if } n=1, \quad \left\{\frac{\Pi(2i-1)}{\Pi i\,\Pi(i-1)}\right\}^2 \text{ if } n=2i, \quad 4\left\{\frac{\Pi(2i-1)}{\Pi i\,\Pi(i-1)}\right\}^2 \text{ if } n=2i+1;$$

thus giving rise to the progression

$$1;\ 1,\ 4,\ 9,\ 36,\ 100,\ 400\ \ldots\ldots$$

Prove also that if, instead of the degrees of U and V in x^2 being *each n*, the degree of their product in x^2 is n, everything else remaining unaltered, the number of solutions is expressed by the square roots of the preceding progression, *i.e.*, is 1; 1, 2, 3, 6, 10, 20, 35 ......

2892. (W. S. McCay, B.A.)—(1) The bicircular quartic

$$Q \equiv lr_1 + mr_2 + nr_3 = 0$$

is the envelope of circles having their centres on the conic

$$C \equiv \frac{l^2}{a\alpha} + \frac{m^2}{b\beta} + \frac{n^2}{c\gamma} = 0,$$

and cutting orthogonally the circle through the foci. Hence derive a construction for points on the curve, and show from it that confocal bicircular quartics cut at right angles.

(2) The quartic Q has its fourth single focus where the conic C meets the circle through the points of reference again. Show that the double foci of the curve are the foci of this conic, and that the conic is a parabola if the curve be a cubic.

2901. (I. H. Turrell.)—A square is divided at random by two straight cuts. Required the probability that one of the pieces is a pentagon.

2904. (Col. Clarke, R.E., F.R.S.)—An infinite number of points are distributed uniformly on the circumference of a circle. If lines be drawn through every pair, what is the law of distribution or density of the points of intersection of these lines?

2905. (The Father of the Fifteen Young Ladies.)—

Mr. Punch's renown
In London town
Brought up in dozens
His country cousins—
Twenty-eight ladies, pretty and shy,
Twenty-one gentlemen, six feet high.

Quoth he, "I invite
Four couples a night,
A belle with a beau,
Whenever you choose;
If only, you know,
Just now, in the session,
You have the discretion
This rule to use—
That never a pair
Of you all shall share
Together twice my evening fare."

Then smil'd and bow'd
The happy crowd
In full content;
And beau with belle,
The hungry sinners,
In eights they went,
And polished off well
Just twenty-one dinners.

They were loth to leave when all was o'er,
And the rule forbade an octave more.
Then went Mr. Punch on:
"I bid you to luncheon,
A beau with a belle,
In couples three;
But look to it well
I never see
Two meet, who have met at table with me."
The joy was loud
Of the happy crowd;
And twenty-eight noons,
In sixes merry,
They plied his spoons
And drank his sherry.

Then, to the fair who alone, as yet,
In his banquet-hall had never met,
He said: "My dears, (it can't be improper,) arrange to go with me all to the opera;
Come only in flocks of pairs never able
To meet in my box or meet at my table."
Then for eight nights,
Oh, all in their best
So charmingly drest,
Came ravishing sights,
In bevies of seven;
And, girt and caress'd
By the dear delights,
Mr. Punch was blest
With peeps at heaven.

Have you the skill
The lists to fill,
And of forty-nine
All pairs combine?

2908. (T. Cotterill, M.A.)—If a, b, c, d be four points on a cubic K having the same tangential n, then a cubic will pass through these five points and the three intersections of a line L with K, having a node at n, the intersections of the conic (n, a, b, c, d) and L being points on the nodal tangents. Find linearly the remaining points of the nodal cubic on the six lines through a, b, c, d.

2916. (I. H. Turrell.)—To construct a triangle geometrically, having given the vertical angle, radius of inscribed circle, and the centre of gravity of the triangle on the circumference of the inscribed circle.

2921. (Professor Sylvester.)—The annexed table

1	1	1	1	1	1	......
1	3	6	10	15	21	......
1	6	20	50	105	196	......
1	10	50	175	496	1176	......
1	15	105	490	1764	....	

is symmetrical, and is obtained as follows :—

The 2nd row by adding up 3 times successively the progression 1, 0, 0, 0, 0, 0,
3rd row ,, 5 times ,, ,, 1, 1, 0, 0, 0, 0,
4th row ,, 7 times ,, ,, 1, 3, 1, 0, 0, 0,
5th row ,, 9 times ,, ,, 1, 6, 6, 1, 0, 0.

In like manner, the 6th row would be got by adding up 11 times successively the progression 1, 10, 20, 10, 1, 0, 0; the 7th row, by adding up 13 times successively the progression 1, 15, 50, 50, 15, 1, 0, 0, and so indefinitely.

Call the number in the rth line and sth column $[r, s]$; then it is required to prove that

$$[r, s] = \frac{\Pi(r+s-1)\,.\,\Pi(r+s-2)}{\Pi(r)\,.\,\Pi(s)\,.\,\Pi(r-1)\,.\,\Pi(s-1)}.$$

The above (r, s) table does for reducible cyclodes in general what the (r, s) table in Question 2886 does for the symmetrical class in particular.

2927. (A. B. Evans, M.A.)—Find the probability that

$$\cos\phi_1 + \cos\phi_2 + \cos\phi_3 > \sqrt{2},$$

where ϕ_1, ϕ_2, ϕ_3 are the angles of an acute-angled triangle.

2937. (H. R. Greer, M.A.)—Relating to Mr. Thompson's empirical theorem (Quest. 2911), and established thereby.

Let Δ be the eccentric angle of the point of contact of the fixed tangent to the exterior ellipse (axes = A, B); let α, β, γ, δ be the eccentric angles of the points of contact of the four tangents drawn as directed—in the pairs α and β, γ and δ—to the inner ellipse (axes = a, b); then

(1) The analytical condition that these should be drawn as geometrically directed is

$$a^2\left[\sin(\alpha+\beta)\sin\gamma\,.\,\sin\delta - \sin(\gamma+\delta)\sin\alpha\,.\,\sin\beta\right]$$
$$= b^2\left[\sin(\alpha+\beta)\cos\gamma\,.\,\cos\delta - \sin(\gamma+\delta)\cos\alpha\,.\,\cos\beta\right].$$

(2) The condition that the four cross-intersections of these pairs of tangents should lie on a circle, is identical with the above.

(3) The equation of this circle is

$$(x^2+y^2)(A^2-a^2)+2xAb^2\cos\Delta+2yaB^2\sin\Delta = A^2b^2+B^2a^2.$$

The centre of this circle lies on the normal at Δ; its radius bears a constant ratio to the central radius vector of the exterior ellipse parallel to the fixed tangent.

2941. (Professor Sylvester.)—Let N_1, N_2 be any two, the same or different, partitionments of n into 3 parts; then it may easily be verified that every combination N_1, N_2 must belong to the one or the other of the 13 subjoined types

$$\begin{array}{cccccccccc} abc & abc & aac & aac & aac & aac & aac & aaa & aaa & aaa \\ \alpha\beta\gamma & \alpha\beta\gamma & \alpha\beta\gamma & \alpha\beta c & \alpha\beta\gamma & \alpha\alpha\gamma & \alpha\gamma\gamma & \alpha\beta\gamma & \alpha\beta\gamma & \alpha\alpha\gamma \end{array}$$

$$\begin{array}{ccc} abc & aac & aaa \\ abc & aac & aaa \end{array}$$

Call the value of any combinations appertaining to these types 24, 20, 12, 10, 8, 6, 4, 4, 2, 2, 7, 1, 0 respectively. Required to prove that the sum of such values for all the combinations possible is $\frac{1}{12}(n-1)(n-2)^2(n-3)$

For example: if $n=6$, the ternarian partitions of 6 are 1, 1, 4; 1, 2, 3; 2, 2, 2; and their binary combinations

$$\begin{array}{cccccc} 1.1.4 & 1.1.4 & 1.1.4 & 1.2.3 & 1.2.3 & 2.2.2 \\ 1.1.4 & 1.2.3 & 2.2.2 & 1.2.3 & 2.2.2 & 2.2.2 \end{array}$$

belong respectively to the types

$$\begin{array}{cccccc} a.a.c & a.a.c & a.a.a & a.b.c & a.a.a & a.a.a \\ a.a.c & a.\beta.\gamma & a.\beta.\gamma & a.b.c & a.\beta.\gamma & a.a.a \end{array}$$

the sum of whose values $= 1+8+2+7+2+0$ is $\frac{1}{12}(5.4^2.3)$.

2943. (W. S. Burnside, M.A.)—All the invariants of a system of three conics are expressible in terms of Θ_{123}, and similar quantities derived by altering the suffixes in every way.

$$\begin{aligned}\Theta_{123} \equiv a_1(b_2c_3+b_3c_2-2f_2f_3)+b_1(c_2a_3+c_3a_2-2g_2g_3)+c_1(a_2b_3+a_3b_3-2h_2h_3)\\ +2f_1(g_2h_3g_3h_2a_2f_3a_3f_2)\\ +2g_1(h_1f_2h_2f_3b_2g_3b_3g_2)\\ +2h_1(f_2g_3f_3g_2c_2h_3c_3h_2),\end{aligned}$$

the equations of the conics being written as usual in the form

$$(a_1b_1c_1f_1g_1h_1)(xyz)^2 = 0, \text{ \&c.}$$

2945. (Lanivic.)—If $n-1 = 6\mu+\nu$, where ν is zero, or any integer less than 6, show that the number of ways of decomposing n into 3 unequal parts is $\mu(3\mu+i-2)$. For example, the number of such parts is 18 if $n = 18$, and 27 if $n = 21$.

2948. (Professor Crofton, F.R.S.)—If 1, 2, 3, 4 are four concyclic points on a Cassinian oval whose foci are F, F′, show that a bicircular oval can be drawn with 1, 2, 3, 4 as foci to pass through F, F′. Show also that the tangents at F, F′ are *double tangents*, each touching the bicircular oval in a second point, and each passing through the centre of the circle 1234.

2958. (Professor Sylvester.)—*Def.* (1). A parallel equality means the equality of the sum of any number of quantities in a group to that of the *same number* of other quantities therein.

Def. (2). The *genus* of a group of quantities is determined by the nature and relations of the parallel equalities it contains, or does not contain.

E.g., A group $[a, b, c, d]$ may be free from parallel equalities, or it may contain them of the form $[a=b]$ or $[a = b,\ c = d]$, or $[a = b = c]$, or $[a=b=c=d]$, or $[a+b = c+d]$, or $[a=b,\ a+b = c+d]$; thus a quaternary group comprises 7 genera.

Determine, according to the same rule, the number of genera in a *quinary* and *sextic* group.

2959. (Professor Sylvester.)—If $U_1, U_2, \ldots U_i$ be respectively the sums of quantics in $x_1, x_2, \ldots x_i$, the lowest quantic forming a term of U_θ being of the degree q_θ, give a short analytical proof that in the equation-system $U_1 = 0,\ U_2 = 0, \ldots U_i = 0$, the trivial solution $x_1 = 0,\ x_2 = 0, \ldots x_i = 0$ will represent $q_1 \cdot q_2 \ldots q_i$ of the total number of solutions; or, in other words, that $q_1 \cdot q_2 \ldots q_i$ will be the multiplicity or intensity of the trivial solution.

2969. (G. A. Ogilvie.)—Show that there is only one point the polars of which with respect to four conics form a parallelogram. Find the area of the parallelogram for any four conics.

2978. (T. Cotterill, M.A.)—Let aa', bb', cc' be fixed points, and conics through aa', cc', and bb', cc' intersect again in p and p', then, if p describe a right line, p' will describe a curve of the sixth order, having triple points at cc', double points at aa', bb', and a single point at the intersection of aa' and bb'. The locus of the points of contact of two such conics which touch is a curve of the fourth order having nodes at cc', and passing through aa', bb'; the envelope of the tangents at these points is a class quartic having the lines through aa' and bb' for double tangents. What do these curves become, if the six points are on the same conic?

3000. (Professor Clifford, F.R.S.)—The circles doubly normal to a bicircular quartic arrange themselves in four systems, each system cutting orthogonally a principal circle. It is required to find the envelope of all the binormal circles of one system.

3001. (A. Martin.)—The bottom of a circular box is covered with an adhesive substance, and two straight rods, each equal in length to the radius of the box, are dropped horizontally into it at random. What is the probability that the rods are crossed in the box?

3008. (W. S. Burnside, M.A.)—Trace the relation between the characteristics of a curve of the mth degree having the maximum number of double points, and the curve enveloped by the line

$$(a_0, a_1, a_2, \ldots a_m)(\theta, 1)^m = 0,$$

where $a_0, a_1, a_2, \ldots a$ are linear functions of the coordinates, and θ a variable parameter.

3013. (Professor Sylvester.)—A body is attracted to a fixed centre by a force varying as any given function of the distance, and perpendicularly towards a fixed plane by a force varying as its distance from the plane: determine the motion.

3014. (Professor Sylvester.)—(1) Show that if n is any integer divisible by 3, n real points may be placed on a cubic curve so as to lie 3 and 3 on $\frac{1}{6}n(n-1)-\frac{1}{3}n+1$ straight lines; *e.g.*, 9 points on 10 lines, 12 points on 19 lines, 15 points on 31 lines, 6 points on 4 lines.

(2) Show also that if n is divisible by 9, n points may be placed on a cubic curve so as to lie 3 and 3 on $\frac{1}{6}n(n-1)-\frac{1}{3}n+3$ straight lines, but that these points cannot all be real.

3019. (Professor Sylvester.)—(1) Prove that the 27 sextactic points in a cubic curve may be resolved into 3 sets of 9 each; the property of each set of 9 being that the line joining any two in the same set will pass through a point of inflexion.

(2) Prove that $6n+3$ points may always be so arranged that each of them shall be the point of intersection of $3n$ right lines obtained by joining two and two all but one couple of the remaining $6n+2$ points. *E.g.*, 15 points may be arranged so that each of them shall be the common intersection of 6 lines drawn through as many pairs of points (*i.e.* 12 points) chosen out of the 14 remaining ones; thus giving rise to $\frac{1}{3}(15\,.\,6)$, *i.e.* 30 right lines in all.

3028. (A. Martin.)—Three points are taken at random, one in each of three faces of a cube. What is the chance that the plane passing through them (1) cuts four faces, (2) that it cuts five faces, and (3) that it cuts all the faces of the cube?

3029. (S. Watson.)—Find the locus of the intersection of normals at the extremities of focal chords in an ellipse.

3055. (R. Tucker, M.A.)—Normals to a lemniscate are drawn at points whose vectorial angles are θ and $\frac{2}{3}\pi+\theta$; find the locus of their intersection, and the area of the curve.

3056. (W. S. Burnside, M.A.)—A binary quantic of the $2n$th degree in x, y may be considered as a ternary quantic of the nth degree in y^2, x^2, $2xy$, having $\frac{1}{2}n(n-1)$ less than the proper number of constants. Prove that, in virtue of this reduction in the number of independent constants, it is possible to arrange the terms, so that any invariant or covariant of the ternary form is an invariant or covariant of the binary form; and moreover, that the additional invariants and covariants of the $2n$-ic are invariants and covariants of the conjoint system, composed of the ternary form of the nth degree, with a ternary quadric.

3057. (B. W. Horne, M.A.)—A bright sphere has its centre at the focus of a paraboloid of revolution; show that the total illumination on the portion of the paraboloid cut off by a plane through the focus perpendicular to the axis is $I\pi(\pi-2)c^2{}_2$, where c is the radius of the sphere.

3059. (Professor Hudson, M.A.)—A right circular cylinder is made of elastic material attached to rigid fixed plane ends. It is distended by fluid pressure. Supposing that the tensions in the meridian and circular sections are regulated by Hooke's law, obtain equations sufficient to determine completely the shape it will assume. If the pressure p be constant, prove that the meridian curve is

$$x+A=\int\frac{\frac{1}{2}py^2+B}{\left\{\left(\frac{\lambda y^2}{2a}-\lambda y+C\right)^2-(\frac{1}{2}py^2+B)^2\right\}^{\frac{1}{2}}}\,dy,$$

where a is the original radius, λ one of the moduli of elasticity, and A, B, C constants of integration. How are they to be determined?

3063. (G. A. Ogilvie.)—In how many cases is it impossible to succeed in getting two packs of 10 cards (numbered from 1 to 10) arranged in order, (1) being allowed to make two packs besides those with which we terminate, (2) being allowed three extra packs? Extend this to the case of an ordinary pack of cards.

3066. (W. S. B. Woolhouse, F.R.A.S.)—Determine a general expression for the average of the *positive* values of the linear function $h + A_1x_1 + A_2x_2 + A_3x_3 + \&c.$, when the variables x_1, x_2, x_3, &c. take all values between given limits, the constants h, A_1, A_2, A_3, &c. having each of them a fixed sign, either positive or negative.

3072. (A. Martin.)—Find (1) the average length of the chord drawn through a given point within a given sphere, (2) the average area of the base of the segment cut off by a plane through the given point, and (3) the average volume of the segment.

3075. (R. Tucker, M.A.)—Ellipses are described having the same major axis, and points are taken on them having the same eccentric angle. If these points are such that the osculating circles at three other points cointersect in them, then the envelope of the circles through the sets of four points is a septic, and the locus of the centres is a cubic curve.

3076. (Rev. T. R. Terry, M.A.)—From a point, at distance a from a centre of force varying as $r^{-2} + 2ar^{-3}$, a particle is projected at an inclination of $\frac{1}{4}\pi$ to the initial distance. Determine the different orbits described, according as the initial velocity bears to that in a circle at the same distance a ratio greater than, equal to, or less than $2 : \sqrt{3}$.

3078. (Professor Hudson, M.A.)—A ray of light traverses a medium in which the density at any point is a function of (r, θ), the polar coordinates of the point; prove that, if μ be the refractive index,

$$\frac{\mu}{\rho} = \frac{\cos\psi}{r}\frac{d\mu}{d\theta} - \sin\psi\frac{d\mu}{dr},$$

where ρ is the radius of curvature of the path of the ray, and ψ the inclination of its tangent to the radius vector.

3079. (C. R. Rippin, M.A.)—Three equal balls A, B, C are placed on a smooth table, so that the centres of A and B are in the straight line which touches the circumference of C; and B is also in contact with a smooth vertical plane to which AB is normal. A fourth equal ball D is made to impinge upon A with velocity u in the line AB. Find the initial motion of each ball; the elasticity of the balls and plane being the same (coefficient e). Also show that whatever be the value of e, the angle between the initial motions of A and C $= \tan^{-1}(\frac{16}{7}\sqrt{3})$.

3084. (A. Martin.)—A cube is cut at random by a plane. What are the respective probabilities that the plane cuts three, four, five, or all of the faces of the cube?

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MUKHOPADHYAY, SYAMADAS, M.A.; Chinsurah.
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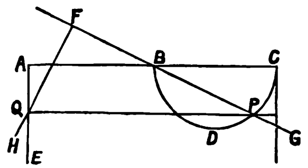
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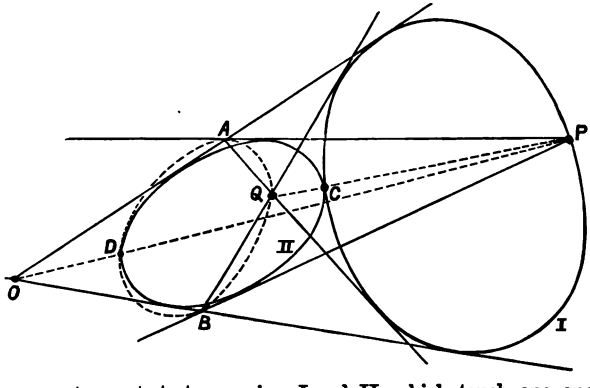
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APPENDIX.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

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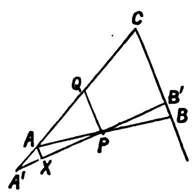
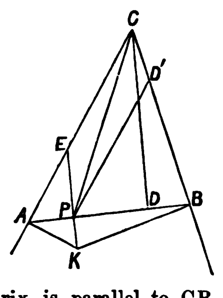
103. (EDITOR.)—Through a given point P within a given angle ACB, draw a straight line such that (1) the triangle ABC shall be a minimum; or (2) the intercepted part APB shall be a minimum.

Solution by (1) W. J. GREENSTREET, M.A., (2) H. J. WOODALL, A.R.C.S.



1. Through A draw APB such that AB is bisected at the given point P; then the triangle ABC will be a minimum. Draw any other secant A′PB′, and draw AX parallel to CB; then the triangle APX = ΔB′PB; thus ΔAA′P > ΔB′PB; hence, adding APB′C to each, ΔA′B′C > ΔABC, and thus the minimum triangle is that whose base is bisected at P.

To obtain the minimum line, draw PQ parallel to CB; take QA = QC; let AP meet CB in B; then, because PQ is parallel to CB, AP/PB = AQ/QC = 1, or AP = PB.



2. Let AB be the least line which passes through P and is terminated by CA, CB. From C draw CD perpendicular to AB. Then AP = DB. [The proof of the proposition of which this is the converse will be found in CASEY's *Sequel*, Book III. 18.]

Draw perpendiculars at A, P, B to AC, AB, CB respectively; these will meet in K (CASEY's *Sequel*). Triangle APK is right-angled at P, and AK is perpendicular to CA. Then K lies on a parabola whose vertex is P, and (by analysis) the directrix is one-fourth the distance from P and parallel to CA. Similarly, by considering the triangle BPK, we find that K lies on a similar parabola whose directrix is parallel to CB. These parabolas meet in P and K; hence we can find K. Then join

KP, and draw APB perpendicular to PK. Make ACP $= \alpha$, PCB $= \beta$, CP $= a$, CPA $= \theta$; then we have

$$u = \text{AB} = \text{AP} + \text{PB} = a \sin(\alpha+\theta)/\sin\alpha + a\sin(\beta+\theta)/\sin\beta$$
$$= a\left[2\cos\theta + \sin\theta(\cot\alpha + \cot\beta)\right];$$

hence for a minimum we shall have

$$\tan\theta = \tfrac{1}{2}(\cot\alpha + \cot\beta),$$

wherefrom we obtain another construction.

[Of the second part (2) of the Question, the following analytical solution is given by Professor LAMPE:—

Draw PD, PE parallel to CA, CB; and put CE $= a$, CD $= b$, EA $= x$, DB $= y$, $\angle$ ACB $= \gamma$, APB (the minimum line) $=$ L; then we have

$$(a+x)^2 + (b+y)^2 - 2(a+x)(b+y)\cos\gamma = (\text{QR})^2, \quad xy = ab,$$

whence $\quad x^4 + 2x^3(a - b\cos\gamma) - x^2(\text{L}^2 + 4ab\cos\gamma - a^2 - b^2)$
$$+ 2abx^2(b - a\cos\gamma) + a^2b^2 = 0;$$

thus, differentiating L^2 and putting $D\text{L}^2/dx = 0$, we find, after dividing by $x+a$, the cubic equation $x^3 - bx^2\cos\gamma + abx\cos\gamma - ab^2 = 0$.

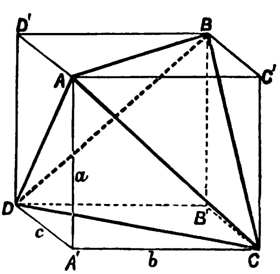
Example.—Let $a=2$, $b=1$, $\gamma=60°$, $(\cos\gamma=\tfrac{1}{2})$; then $x^3 - \tfrac{1}{2}x^2 + x - 2 = 0$;

$$\therefore \quad x = 1{\cdot}1474738, \quad \text{L} = 2{\cdot}965978.$$

The cubic equation for x^2 may easily be proved to possess always one real root between b and a, if a is supposed to be greater than b.]

12032. (Professor MORLEY.)—A tetrahedron, whose opposite edges are equal, is treated conveniently by considering four corners of a rectangular box, no two of which form an edge of the box (CAYLEY's *Collected Papers*, Vol. v., p. 559). There are five spheres which touch the faces of such a tetrahedron. Prove that their centres are the centre of the box and its four remaining corners.

Solution by Professors DROZ-FARNY, BHATTACHARYA; *and others.*



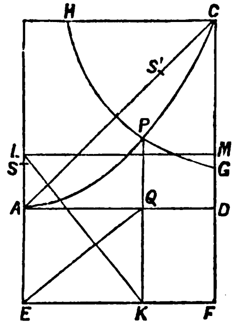
Dans tout parallélipipède rectangle les quatre diagonales sont égales et se coupent en un point O en parties égales. Ce point est le centre de la sphère circonscrite au parallélipipède donc au tétraèdre isoscèle ABCD. Les faces du tétraèdre étant égales, leurs circonférences circonscrites sont égales, et par conséquent à égale distance du centre O, qui est donc aussi le centre de la sphère inscrite au tétraèdre.

En représentant par a, b, c les arêtes du tetraèdre trirectangle A′CAD on sait que la perpendiculaire abaissée de A′ sur le plan CAD est fournie par la relation $1/h^2 = 1/a^2 + 1/b^2 + 1/c^2$.

Il est facile de démontrer que les perpendiculaires abaissées de A′ sur les trois autres faces du tetraèdre ABCD sont égales à h. En effet la droite A′C′ étant divisée en deux parties égales par le plan ABC les perpendiculaires abaissées de A′ et de C′ sur ce plan sont égales.

Or le tétraèdre C′ABC étant égal à A′ACD la perpendiculaire de C sur ABC $= h$.

12169. (D. Biddle.)—Let ABCD be a square, of side = unity. Produce BA, making AE = any value $\not> 1$, and complete the rectangle AEFD. With focus S, on AB, at $\frac{1}{4}$AB from A, describe the parabola APC; and with focus S′, on the diagonal AC, at 2AE from A, describe the hyperbola GPH, having AB, AD (produced) as its asymptotes. From the point P, in which the two curves intersect, draw PQ perpendicular to AD, and produce to K. Join QE, and draw KL at right angles to it, cutting AB in L. Finally draw LM parallel (and equal) to AD. Prove (1) that EL, EK (=AQ) are respectively $AE^{\frac{1}{3}}$, $AE^{\frac{2}{3}}$; (2) that PQ, QK, KE, EL, LM are consecutive terms in a geometrical series; (3) that by a process like that herein set forth, it is possible, if we have the nth and the $(n+3)$rd terms, to fill in, to any required extent, the remaining terms of any geometrical series; and (4) that the extraction of the cube-root of any value by geometric methods is a very simple matter, provided Euclid's postulate regarding the circle be extended to the other conic sections, as in justice it should be.



Solution by the Proposer.

(1) Since P is on the particular parabola $PQ = AQ^2$, and since P is on the particular hyperbola

$$PQ \cdot AQ = \tfrac{1}{4}(AS')^2 = AE^2,$$

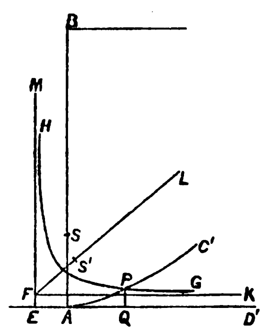
$$\therefore \quad AQ = AE^{\frac{2}{3}}, \text{ and } PQ = AE^{\frac{4}{3}}.$$

Again, by construction, QKE, KEL are similar triangles, and

$$QK : KE = KE : EL;$$

$$\therefore \quad EL = KE^2/QK = AQ^2/AE = AE^{\frac{1}{3}}.$$

(2) PQ, QK, KE, EL, LM are respectively $AE^{\frac{4}{3}}$, $AE^{\frac{3}{3}}$, $AE^{\frac{2}{3}}$, $AE^{\frac{1}{3}}$, $AE^{\frac{0}{3}}$, and are therefore in geometrical progression, with a ratio $= AE^{-\frac{1}{3}}$.



(3) Originally the 2nd and 5th terms alone were given; but now the series can be indefinitely extended in either direction.

(4) The parabola and hyberbola here brought into requisition are not only absolutely defined, but can practically be drawn with an accuracy not inferior to that with which a circle is ordinarily produced. Moreover, the cube root of any value can be found by the method, for AB : AE may be any ratio, and AB : EL : AQ : AE will always form a geometric series, either extremity of which may be taken as unity. Thus

$$\text{AB}(=1) : \text{AE}^{\frac{1}{3}} : \text{AE}^{\frac{2}{3}} : \text{AE}(<1), \text{ or } \text{AB}(>1) : \text{AB}^{\frac{2}{3}} : \text{AB}^{\frac{1}{3}} : \text{AE}(=1).$$

In the latter case, AQ is the cube root.

By an extension of the process set forth in this question, we obtain a graphic method of solving cubic equations.

The equation $x^3+px^2+qx+r=0$ becomes

$$\frac{x+p}{r} = -\left(\frac{1}{x^2}+\frac{q}{rx}\right), \quad \left(\frac{q^2-4pr}{4r^2}\right)-\frac{x}{r} = \left(\frac{1}{x}+\frac{q}{2r}\right)^2 \text{......} (\alpha),$$

or

$$y = z^2;$$

also

$$\frac{x}{r} = \frac{q^2-4pr}{4r^2}-y, \quad \frac{1}{x} = z-\frac{q}{2r} \text{ ....} (\beta, \gamma),$$

and

$$\frac{x}{r} \cdot \frac{1}{x} = \frac{1}{r} = \frac{1}{4}\left(\frac{2}{\sqrt{r}}\right)^2 \text{ ....} (\delta).$$

Let AB be the side (= unity) of the square ABCD. With the focus S at $\frac{1}{4}$AB from A, and with a directrix an equal distance the other side of AD, describe the parabola APC. This gives us the locus indicated in (α), especially if carried through A on the other side of AB as well. But subsequently, eight cases arise, according to the several signs of r, q^2-4pr, and $q/(2r)$.

Taking the case in which the first and third of these values are negative and the second positive, produce DA to E, making AE $= q/(2r)$, and draw EM at right angles, taking in it EF $= (q^2-4pr)/(4r^2)$. Draw FK parallel to AD, and bisect the right angle MFK by FL. Upon this, take the focus S′ at a distance from F, $= 2/\sqrt{r}$, and describe the hyperbola GPH, having FM, FK as its asymptotes. From P, the point of intersection of the two curves, draw PQ perpendicular to AD. Then it is easy to see that PQ $= y$, AQ $= z$, EQ $= 1/x$, and AB/EQ $= x$, a real root of the original cubic equation.

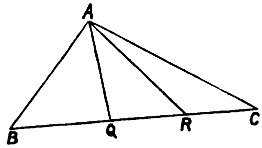
A quadratic can now be formed, $x^2+(p+\text{AB/EQ})\,x-r\text{EQ/AB} = 0$, by which to find the remaining roots, real or imaginary.

Of course, in other cases, E and F may be on the other side of AB, and F also on the other side of ED. Moreover, FL may run parallel to BD (instead of AC), as when (β) maintains its present form after reduction of signs.

1254. (Editor.)—If the base BC of a triangle ABC be trisected in Q, R, prove that $\sin \text{BAR} \cdot \sin \text{CAQ} = 4 \cdot \sin \text{BAQ} \cdot \sin \text{CAR}$ ...... (1);

and $(\cot \text{BAQ} + \cot \text{QAR})(\cot \text{CAR} + \cot \text{RAQ}) = 4 \operatorname{cosec}^2 \text{QAR}$ ... (2).

Solution by R. CHARTRES; Professor KRISHNAMACHARRY; *and others.*



Since $\quad BR \,.\, CQ = 4BQ \,.\, CR,$

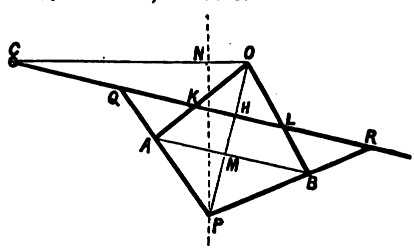
$\sin BAR \,.\, \sin CAQ$

$= 4 \sin BAQ \,.\, \sin CAR \ldots (1)$

or $\dfrac{\sin BAR}{\sin BAQ \,.\, \sin QAR} \cdot \dfrac{\sin CAQ}{\sin CAR \,.\, \sin QAR} = \dfrac{4}{\sin^2 QAR}$, which is (2).

6491. (E. B. ELLIOTT, M.A.)—OA, OB are two equal rods capable of revolving independently in the same plane round a common fixed extremity O. The other extremities A, B, are freely jointed each to the middle point of one of two other equal rods PQ, PR, freely jointed at P a common extremity, and having their other ends constrained by a straight groove to move along another rod CQR, which has one point C fixed and can revolve about it. Show that, as the system moves, P describes a straight line.

Solution by H. J. WOODALL, A.R.C.S.



In the figure denote OA, PQ, OC, OP by a, b, c, r respectively.

Let $\quad COP = \phi$

then $\quad OH = c \cos\phi,$

$HP = r - c\cos\phi,$

$\cos APM = (r - c\cos\phi)/b,$

$a^2 = AO^2$

$= AP^2 + PO^2 - 2AP \,.\, PO \cos APO = \frac{1}{4}b^2 + r^2 - r\,(r - c\cos\phi)$

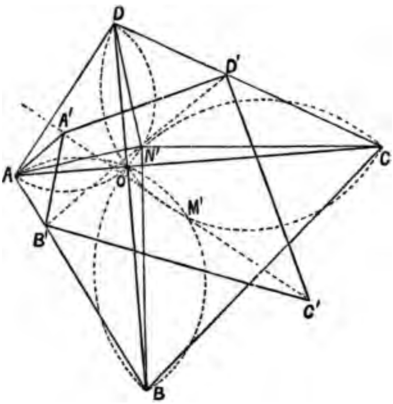
$= \frac{1}{4}b^2 + rc\cos\phi\,; \qquad \therefore \quad r\cos\phi = (a^2 - \frac{1}{4}b^2)/c.$

[The locus of P is the straight line PN drawn perpendicular to CO, and generally, if PA : PQ = PB : PR = n : 1 (n being whole or fractional), P will describe a straight line PN perpendicular to OC, where

$$ON = (a^2 - n^2b^2)/c.]$$

12090. (Professor NEUBERG.)—On donne, dans un même plan, deux droites AC, BD égales et perpendiculaires entre elles. Démontrer que l'on peut construire une infinité de quadrilatères A′B′C′D′, tels que les points A, B, C, D sont les centres des carrés construits extérieurement sur les côtés A′B′, B′C′, C′D′, D′A′; les points A′, B′, C′, D′ décrivent des figures égales.

Solution by Prof. DROZ-FARNY; H. W. CURJEL, B.A.; *and others.*

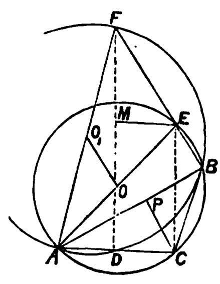


On démontre aisément le lemme suivant : les deux droites qui joignent le milieu d'un côté d'un triangle aux centres des carrés construits extérieurement sur les deux autres côtés sont égales et perpendiculaires entre elles.

Soit A′B′C′D′ un des quadrilatères cherchés, N′ le milieu de la diagonale B′D′, et M′ celui de la diagonale A′C′, enfin O le point d'intersection de AC et BD. D'après le lemme les circonférences décrites sur AB et CD se croisent en O et M′; de même les circonférences décrites sur BC et AD comme diamètres se coupent en O et N′. Les points M′ et N′ sont donc bien déterminés. Prenons A′ arbitrairement; faisons angle A′AB′ = 90°; puis A′A = AB′. C′ sera le symétrique de A′ par rapport à N′ et D′ le symétrique de B′ par rapport à N′. Il résulte de la construction même que les points A′, B′, C′, et D′ décrivent des figures égales.

12073. (M. BRIERLEY.)—ABC is a plane triangle, obtuse at C, inscribed in the circle ACBE, perpendicular to the base AB; draw BE meeting the circle in E, and produce it so that EF = AC. Through the points A, B, F describe another circle intercepting AC in D. Then will the distance of the centres of the two circles be equal to ½AC, and CD will be equal to the perpendicular from C upon the base AB.

Solution by T. SAVAGE; W. J. GREENSTREET, M.A.; *and others.*



Since ∠B is right, AE and AF are diameters of the circles ABE, ABF respectively. Hence join of centres of these circles is parallel to EF and equal ½EF, that is, equal ½AC.

Again, join F, D; E, C; draw EM, CP perpendicular to FD, AB respectively. Angles D, and ECD are (III. 31) right. Hence CM is a rectangle; therefore

EM = CD.

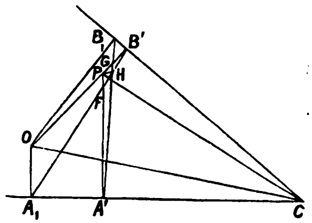
But ∠BAD = ∠BFD;

and AC = EF;

therefore EM = CP, therefore CD = CP.

9858. (N'Importe.)—On considère les pieds A′B′C′ des hauteurs et les pieds A_1, B_1, C_1 des médianes d'un triangle ABC. Démontrer que les droites $A'B_1$ et A_1B', $B'C_1$ et B_1C', $C'A_1$ et C_1A' se coupent deux à deux sur la droite qui joint le point de concours des hauteurs au centre du cercle circonscrit au triangle.

Solution by H. W. Curjel, B.A.



Let O and P be any points isogonally conjugate with respect to the angle C. Draw perpendiculars OA_1, PA'; OB_1, PB' to the lines containing the angle C. Let $B'A_1$, B_1A' meet in H. Let $B'A_1$ cut PA′ in F, and B_1A' cut PB′ in G. The figures PA′CB′, OB_1CA_1 are evidently similar, and

$$A'C : CB_1 = B'C : CA_1;$$

$\therefore$ triangles $A'B_1C$, $B'A_1C$ are similar;

$$\therefore \angle PA'G = \angle PB'F;$$

$\therefore$ triangles PA′G, PB′F are similar;

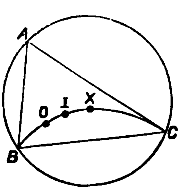
$$\therefore \quad FP : PG = PB' : PA' = OA_1 : OB_1.$$

Hence A_1F, B_1G cut OP in the same point. That is, A lies on OP.

Hence, if O, P are two points isogonally conjugate with respect to a triangle ABC, and A_1, B_1, C_1 are the feet of the perpendiculars from O on the sides of the triangle, and A′, B′, C′ the feet of the perpendiculars from P on the sides of the triangle, $A'B_1$ and A_1B', $B'C_1$ and B_1C', $C'A_1$ and C_1A' cut two and two on OP. The theorem of the question is a particular case of this, since the orthocentre and circumcentre are isogonally conjugate.

11921. (R. Chartres.)—BC is a fixed straight line; find the locus of a point A, when the circumcentre, the in-centre, and the ortho-centre of the triangle ABC lie on the same curve, and show that a fourth point, besides B and C, also lies on the curve.

Solution by the Proposer; Prof. Bhattacharya; *and others.*

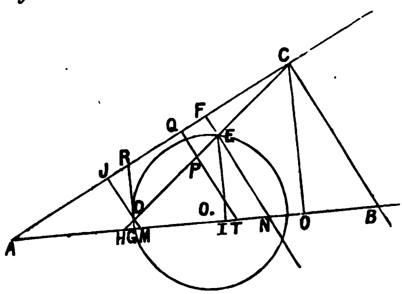


Let X, I, O be the circumcentre, the in-centre, and the ortho-centre of the triangle ABC, then the angles subtended by BC at X, I, O are 2A, $90° + \frac{1}{2}A$, $180° - A$ respectively. These are all satisfied by $A = 60°$.

Hence, if A describe an arc, angle $= 60°$, X, I, O will lie on the supplemental arc, angle $= 120°$; and Fermat's point will evidently lie on the same arc.

1015. (M. Collins, M.A.)—Required the position of the straight line AB, intersecting at A and B two straight lines CA, CB given in position, and touching a fixed circle ODE at P; so that, if CD be perpendicular to AB, and Q a given point, AD shall have to QB a given ratio.

Solution by Morgan Brierley.



Through P draw TPQ perpendicular to AC, and make AQ : AJ (in AC) equal to the given ratio of the required perpendiculars upon AB and AC. From J draw JDM perpendicular to AC, cutting the circle in D, and AB in M, and through D and P draw the chord HDPEC, intercepting AB, AC in H and C. From E draw EI perpendicular to AB; then EI : DJ = AQ : DJ, the given ratio.

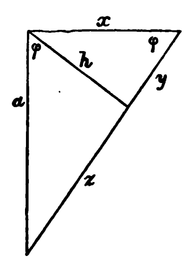
Through D, E draw the perpendiculars RG to AB, and FN to AC; also CO perpendicular to AB; then, by similar triangles,

$$\text{EI} : \text{DJ} = \text{DR} : \text{EN} = \text{CO} : \text{CB} = \text{AC} : \text{AB} = \text{AQ} : \text{AT} = \text{AJ} : \text{AM};$$

$$\therefore \quad \text{AM} : \text{AT} = \text{AJ} : \text{AQ} = \text{EI} : \text{DJ}.$$

1239. (Editor.)—Given one side of a right-angled triangle; construct it, so that the difference between the other side and the adjacent segment of the hypotenuse, cut off by a perpendicular from the right angle, may be a *maximum*. Prove that the perpendicular divides the hypotenuse in extreme and mean ratio, and that the greatest segment is equal to the remote side of the triangle.

Solution by Professor Lampe.



We have

$$h = a\cos\phi,\quad z = a\sin\phi,\quad x = a\cot\phi,$$
$$y = a\cot\phi\cos\phi,$$

whence $\quad x - y = f(\phi) = a\cot\phi\,(1-\cos\phi).$

$$f'(\phi) = a\left(\cos\phi - \frac{1}{1+\cos\phi}\right),$$

$$f''(\phi) = -a\left(\sin\phi + \frac{\sin\phi}{(1+\cos\phi)^2}\right).$$

$f'(\phi) = 0$ furnishes $\cos\phi = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$, an angle easy to construct. Calculating x, we

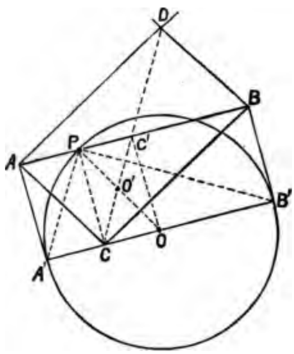
obtain from $\cos^2\phi + \cos\phi - 1 = 0 \quad \cos\phi = \sin^2\phi$,

$$x = \frac{a\cos\phi}{\sin\phi} = \frac{a\cos\phi \,.\, \sin\phi}{\sin^2\phi} = a\sin\phi = z.$$

At last, having always $x^2 = y\,(y+z)$, and $z = x$, we get $y : z = z : y + z$.

12065. (Professor BARISIEN.)—Si par le point de rebroussement d'une cardioïde on mène trois droites inclinées l'une sur l'autre de 60°, ces droites rencontrent chacune la cardioïde en deux points autres que le point de rebroussement. Les tangentes à la courbe en ces deux points sont rectangulaires et leur points d'intersection se trouvent sur un cercle fixe.

Solution by Profs. DROZ-FARNY, CHAKRIVARTI, *and others.*



On sait que la podaire d'un cercle par rapport à un point P de sa circonférence est une cardioïde ayant P comme point de rebroussement. Soit donc AB une sécante quelconque passant par P et A′B′ le diamètre du cercle générateur qui lui est parallèle. Les tangentes au cercle en A′ et B′ coupent évidemment la sécante en deux points A et B de la cardioïde. Soit C la projection de P sur A′B′. Les droites AC et BC, qui divisent respectivement PA′ et PB′ en parties égales, sont les normales en A et B à la courbe.

Or évidemment

angle ACB = A′PB′ = 90°.

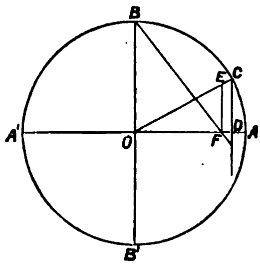
Les tangentes AD et BD se coupent a angle droit et la figure ABCD est un rectangle. Comme on le voit C′, point de coupe des diagonales AB et CD, est la projection du centre O sur la sécante AB.

Par conséquent CD passe par le point fixe O′ milieu de PO et O′D = 3O′C. Or le lieu de C est la circonférence sur PO comme diamètre. Le lieu de D est donc aussi une circonférence concentrique de rayon triple. La restriction de trois droites à 60° est inutile.

11923. (Professor DECAMPS.)—Soient AA′, BB′ deux diamètres rectangulaires d'un cercle O. D'un point quelconque C de la circonférence, on abaisse une perpendiculaire CD sur AA′; on porte sur OC une longueur

OE = OD, et l'on projette E en F sur AA'. Trouver le lieu du point de rencontre des droites BF et CD. La figure donne le moyen de construire un angle dont la tangente soit le carré du cosinus d'un angle donné, ou dont le cosinus soit la racine carrée de la tangente d'un angle donné.

Solution by H. W. Curjel, B.A.; Prof. Mukhopadhyay; *and others.*



Take OA, OB as axes of x and y.

Let $\angle COA = \theta$.

Equation to CD is $x = \cos\theta$;

equation to BF is $x/(a\cos^2\theta) + y/a = 1$;

$\therefore$ equation to locus is $xy - xa + a^2 = 0$.

It is evident from the figure that

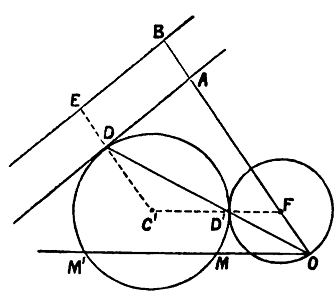
$$\tan FBO = \cos^2 COA.$$

If the $\angle$ FBO is given, the point E may be determined as the intersection of a circle on OA as diameter and FE, which is perpendicular to OA.

The E determines the angle COA, whose cosine is the square root of tan FBO.

12062. (Professor Rebuffel.)—Deux points mobiles M et M' se déplacent sur une droite XY de façon que le produit OM . OM' de leurs distances à un point fixe O de cette droite soit constant; (1) si, par les points M, M', et un point fixe P, on fait passer une circonférence C, trouver le lieu de son centre, et (2) si par M et M' on fait passer une circonférence C' tangente à une droite fixe D, trouver le lieu de son centre.

Solution by Professors Droz-Farny, Sarkar, *and others.*



1. OP coupe la circonférence PMM' en P'; on a alors

$$OM . OM' = OP . OP' = \text{const.}$$

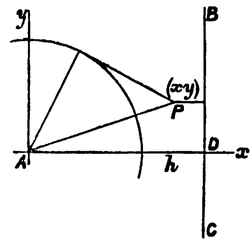
Le point P' est donc fixe aussi, et le lieu cherché est la perpendiculaire sur PP' en son point milieu.

2. Abaissons de O une perpendiculaire OA sur D. Soit C' une des circonférences tangentes à D. Transformons la figure par inversion, le module étant OM . OM'. La circonférence C' se transforme en elle-même; la droite D a pour inverse une circonférence de centre F sur OA et tangente à C' au point D' correspondant à D. Or la circonférence F est fixe. Prolongeons OA d'une longueur AB égale au rayon

FD′; on aura C′D = C′D′, DE = D′F, C′E = C′F. Le lieu cherché est une parabole admettant F comme foyer et la droite BE comme directrice.

2775. (J. Wilson, M.A.)—Prove that the locus of a point the square of the tangent from which to a fixed circle varies as its distance from a fixed line, is a circle which cuts, does not cut, or touches, the fixed line, according as it cuts the circle, passes without it, or touches it.

Solution by R. Chartres.



Let the axis of y be parallel to the given straight line BC, and let AD = h then the) locus of P is

$$(x^2+y^2-a^2)+\lambda(x-h) = 0,$$

a circle which passes through the intersection of $x^2+y^2=a^2$, and $x=h$, and thus satisfies the problem.

11683. (Rev. Robert Bruce, D.D.)—Show (1) how to place eight men on a draught-board so that no two of them shall be in line with one another, horizontally, perpendicularly, or diagonally; and find (2) in how many ways this can be done.

Solution by Professor Nash.

The answer to this question given on p. 31 of Vol. LIX. does not give the number of ways in which eight queens can be placed on a chess-board so that no two of them can take each other. By actual trial, I find that there are twelve distinct ways of placing them, the following the numbers of the squares occupied:—

(1)	1,	13,	24,	30,	35,	47,	50,	60;
(2)	1,	14,	24,	27,	39,	44,	50,	61;
(3)	2,	12,	22,	32,	35,	41,	55,	61;
(4)	2,	13,	23,	25,	35,	48,	54,	60;
(5)	2,	13,	23,	28,	33,	48,	54,	59;
(6)	2,	14,	17,	31,	36,	48,	51,	61;
(7)	2,	14,	24,	27,	33,	44,	55,	61;
(8)	2,	15,	19,	30,	40,	45,	49,	60;
(9)	2,	15,	21,	32,	33,	44,	54,	59;
(10)	2,	13,	24,	28,	33,	47,	50,	62;
(11)	3,	14,	24,	25,	37,	47,	50,	60;
(12)	3,	13,	18,	32,	33,	47,	52,	62.

Since any one of the four corner squares may be taken as the first square, and either of the two adjoining squares as the second, each of these twelve gives rise to eight methods of placing the pieces, but in the case of the last these are the same in pairs, so that the total number of arrangements is 92.

One of the conditions of the question is that one piece must be placed in each row, and one in each column; this gives the necessary (but not sufficient) condition that the sum of the numbers must be 260, this number being the sum of the two Arithmetical Progessions $8+16+\ldots+56$ and $1+2+3+\ldots+8$.

I do not think there is any other mathematical principle involved in the problem, though of course it would be possible to state it as a question of pure arithmetic, without any reference to the chess-board. The knight's move does not appear to have any direct connexion with the question, though it naturally occurs very frequently, since this move represents the shortest distance between two queens that do not attack each other. There is no square on the board that is not included in several of the ninety-two arrangements, so that one queen can be placed at random on the board.

1090. (E. Harrison, M.A.)—In a right-angled triangle inscribe a square, one of its angles coinciding with the right angle of the triangle; in the two right-angled triangles thus formed inscribe in the same way squares; in the four right-angled triangles now formed inscribe as above squares; and continue this process; after the nth operation there will be 2^n squares. Prove that the perimeters of these 2^n squares are equal to the perimeter of the first square.

Solution by Profs. Lampe, Mukhopadhyay, *and others.*

Let a and b be the sides of the right-angled triangle, and put $a/(a+b)=\alpha$, $b/(a+b)=\beta$. Then, after inscribing the first square, we get two triangles with the sides (1) $a\alpha$, $b\alpha$; (2) $a\beta$, $b\beta$.

Repeating the construction with both triangles, we obtain four triangles, with sides

$$(1)\ a\alpha^2,\ b\alpha^2;\quad (2)\ a\alpha\beta,\ b\alpha\beta;\quad (3)\ a\alpha\beta,\ b\alpha\beta;\quad (4)\ a\beta^2,\ b\beta^2;$$

and so on. Every new series of sides results from the preceding by multiplying them at first with α, then with β. Thus the last series will be

$$a\alpha^n,\ b\alpha^n;\quad \binom{n}{1} \text{ times } a\alpha^{n-1}\beta,\ b\alpha^{n-1}\beta;\quad \binom{n}{2} \text{ times } a\alpha^{n-2}\beta,\ b\alpha^{n-2}\beta;\ \&c.$$

The square inscribed in the first triangle has the side $ab/(a+b)$; those inscribed in the last series of triangles $\dfrac{ab\alpha^n}{a+b}$, $\dfrac{ab\alpha^{n-1}\beta}{a+b}$, &c.

Their sum will be evidently $\dfrac{ab}{a+b}(\alpha+\beta)^n=\dfrac{ab}{a+b}$.

12099. (Professor BARISIEN.)—Le lieu des points tels qu'en abaissant les quatre normales à une ellipse, la somme des carrés des six distances mutuelles des pieds des normales soit constante, est une conique. Dans quel cas ce lieu se compose-t-il de lignes droites ?

Solution by Professors SCHOUTE, DROZ-FARNY, *and others.*

In the point $a\cos\phi$, $b\sin\phi$ of the ellipse $x^2/a^2+y^2/b^2=1$, the normal is $ax\sin\phi-by\cos\phi=c^2\cos\phi\sin\phi$, an equation of the fourth degree in $\sin\phi$ or $\cos\phi$, if x, y are the coordinates of a given point. Moreover, the condition is $\Sigma a^2(\cos\phi_i-\cos\phi_j)^2+\Sigma b^2(\sin\phi_i-\sin\phi_j)^2=k^2$.

By elimination of the symetric functions of the four quantities $\sin\phi$, and the four quantities $\cos\phi$, we find for the equation of the locus

$$a^2(a^2-2b^2)x^2+b^2(b^2-2a^2)y^2=\left\{k^2-2(a^2+b^2)\right\}c^4,$$

where c^2 stands for a^2-b^2. This conic is an ellipse for $2b^2>a^2>b^2$ (real for $k^2<2a^2+2b^2$), and a hyperbola for $a^2>2b^2$. It consists of two parallel lines for $a=b\sqrt{2}$ (real for $k<b\sqrt{6}$) and of two intersecting lines for $k^2=2(a^2+b^2)$, &c.

2637. (Professor WHITWORTH.)—If ϵ_r^x denote the sum of the series obtained by expanding ϵ^x in ascending powers of x as far as the term involving x^r inclusive, then

$$1=\epsilon_n^{-1}+\frac{\epsilon_{n-1}^{-1}}{1}+\frac{\epsilon_{n-2}^{-1}}{1\,.\,2}+\frac{\epsilon_{n-3}^{-1}}{1\,.\,2\,.\,3}+\ldots+\frac{\epsilon_0^{-1}}{n!}.$$

Solution by R. CHARTRES; Prof. CHAKRIVARTI; *and others.*

The dexter is the following right-angled expression, whereof the hypothenuses vanish, being the coefficients of the different powers of x in the expansion of $(e^{-x}\times e^x)$; hence the result $=1$.

$$\begin{array}{llllll}
1 & -1 & +\frac{1}{2!} & -\frac{1}{3!} & \ldots & +\frac{1}{n!}\\[4pt]
1 & -1 & +\frac{1}{2!} & \ldots & -\frac{1}{(n-1)!} & \\[4pt]
\frac{1}{2!} & -\frac{1}{2!} & \ldots & +\frac{1}{2!\,(n-2)!} & & \\[4pt]
\frac{1}{3!} & \ldots & \frac{1}{3!\,(n-3)!} & & & \\[4pt]
\ldots-\frac{1}{(n-1)!} & & & & & \\[4pt]
\frac{1}{n!} & & & & &
\end{array}$$

1038. (S. WATSON.)—A straight line always cuts off a given area from a given parabola; find the curve which it always touches.

Solution by Professor LAMPE.

"The tangent to the interior of two similar, similarly placed, and concentric conics cuts off a constant area from the exterior conic" (SALMON'S *Conics*, § 396). The curve required is therefore a parabola of the same size, with the same axis, only carried along this axis for a certain distance.

6493. (C. LEUDESDORF, M.A.)—If two conics

$$(a, b, c, f, g, h)(x, y, 1)^2 = 0, \text{ and } (a', b', c', f', g', h')(x, y, 1)^2 = 0$$

are such that their four points of intersection lie on a conic passing through their centres, prove that

$$d/dc\,(\log \Delta)\; d/dc'\,(\log \Delta') = d/dc\,(\Theta/\Delta)\; d/dc'\,(\Theta'/\Delta').$$

Find to what this reduces when the conics are both circles.

Solution by H. J. WOODALL, A.R.C.S.

The conic passing through the four points of intersection of the given conics is $S \equiv S_1 - \lambda S_2 = 0$; if this passes through

$$\{(hf-bg)/(ab-h^2),\ (gh-af)/(ab-h^2)\}$$

(the centre of 1), we have, by substituting this in S, and multiplying through by $(d\Delta/dc)^2 = (ab-h^2)^2$,

$$\Delta\, d\Delta/dc = \Delta^2\, d/dc\,(\log \Delta) = -\lambda\Delta^2\, d/dc\,(\Theta/\Delta) \ldots\ldots\ldots\ldots\ldots(3);$$

so, substituting centre of (2), we get

$$\lambda\Delta'\, d\Delta'/dc' = \lambda\Delta'^2\, d/dc'\,(\log \Delta') = -\Delta'^2\, d/dc'\,(\Theta'/\Delta') \ldots\ldots\ldots(4),$$

whence $d/dc\,(\log \Delta)\, d/dc'\,(\log \Delta') = d/dc\,(\Theta/\Delta)\, d/dc'\,(\Theta'/\Delta') \ldots\ldots\ldots(5).$

If the conics are both circles,

$a = b,\ h = 0,\ a' = b',\ h' = 0$, and we get

$$a\,(ac-f^2-g^2) = \lambda\,\{a'\,(f^2+g^2) + c'a^2 - 2a\,(ff'+gg')\} \ \ldots\ldots(3)',$$

$$\lambda a'\,(a'c'-f'^2-g'^2) = a\,(f'^2+g'^2) + a'^2c - 2a'\,(ff'+gg') \ \ldots\ldots(4)';$$

and therefore

$$a\,(ac-f^2-g^2)\,\{a\,(f'^2-g'^2) - 2a'\,(ff'+gg') + a'^2c\}$$
$$= a'\,(a'c'-f'^2-g'^2)\,\{a'\,(f^2+g^2) - 2a\,(ff'+gg') + a^2c'\}\ldots(5)'.$$

1267. (EDITOR.)—If n letters were put at random into n envelopes, already properly addressed, find the probability that no letter would be put into the right envelope.

Solution by Professor LAMPE.

The question is essentially the same as Quests. 7227 (see Vol. XL., pp. 22–24) and 7715 (Vol. LIII., p. 68); and the probability required is

$$\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^n\frac{1}{n!}.$$

This problem is one of the first questions treated by the methods of the calculus of probability. Emerging at periodical intervals, it seems to possess peculiar charms. PIERRE REMOND DE MONTFORT (1678–1719) published in 1708 the work *Essai sur les Jeux de Hazard*, where he treats, in the second part, the game the French call "Treize," which is our problem. MONTFORT's proof is due to NICOLAUS BERNOULLI. In 1884, Professor SANIO published in HOPPE's *Archiv für Math.*, Vol. LXX., p. 224, a question concerning a combinatory definition of the number e, equivalent to our problem. In a note inserted on p. 439 of the same volume, I referred to the old questions of probability connected with it, and pointed out a passage in LAPLACE's *Théorie analytique des Probabilités*, where the problem is treated in all its generality. Given to the competitors for a place in the seminary for students of mathematics at the University of Berlin in 1862, it had the following form:—If once upon a time the Διάβολος should throw all things asunder, what is the probability that nothing shall be left in its right place? This has been reproposed as a new Question, No. 12184.

12088. (Professor SYLVESTER.)—Prove that, for complex numbers $a+b\rho$, where $\rho^2+\rho+1=0$, 2 is a quadratic residue for all prime real integers of the form $24i+5$.

Example: $(1-3\rho)^2 = 1-6\rho+9\rho^2 = -8-15\rho \equiv 2 \pmod 5$.

Solution by H. W. CURJEL, B.A.

Let $p\ (=24i+5)$ be a prime. The congruence $(a+b\rho)^2 \equiv 2 \pmod p$ is satisfied if

$$a^2-b^2+(2a-b)\,\rho b \equiv 2 \pmod p \quad \ldots\ldots\ldots\ldots\ldots\ldots(1);$$

therefore if $a^2-b^2\equiv 2$ and $b\equiv 2a \pmod p$. [Evidently $b\equiv 0 \bmod p$ will not satisfy (1), for $(2/p) = (-1)^{\frac{1}{8}(p^2-1)} = -1$.] Therefore if $b\equiv 2a$, and $3a^2\equiv -2\equiv 24i+3 \pmod p$, *i.e.*, if $a^2\equiv 8i+1 \pmod p$.

Now $(3/p)(8i+1/p) = (-2/p) = (-1/p)(2/p) = (-1)^{p-1}(-1)^{\frac{1}{8}(p^2-1)} = -1$;

but $$(3/p)(p/3) = (-1)^{\frac{1}{4}(3-1)(p-1)} = 1,$$

and $(p/3) = (24i+5/3) = (\tfrac{2}{3}) = -1$, $\therefore (3/p) = -1$, $\therefore (8i+1/p) = 1$;

i.e., $8i+1$ is a quadratic residue of p. Hence theorem.

11001. (Professor MORLEY.)—Let the Jacobian points of the triangle a_1, a_2, a_3 be j_1, j_2, j_3 (so that $a_1, j_3, a_2, j_1, a_3, j_2$ form a harmonic hexagon).

Let a, j be the centroids of the two triangles, and let m_r be the mean point of a_r, j_r $(r = 1, 2, 3)$. Prove that a, j are foci of the maximum in-ellipse of the triangle $m_1m_2m_3$.

Solution by D. P. HIBBERD; Professor BEYENS; *and others.*

Let a be the origin; then $a = \frac{1}{3}(a_1+a_2+a_3) = 0$, $j = \frac{1}{3}(j_1+j_2+j_3)$,

$$m_r = \tfrac{1}{2}(a_r+j_r), \quad r = 1, 2, 3.$$

If a is a focus of the maximum inscribed ellipse, we have the well-known relation between the vectors from the focus to the vertices of the triangle m_r, (1) $\Sigma \dfrac{2}{a_r+j_r} = 0$. This relation will now be proved.

We know a_1j_1 to be harmonic to a_2a_3, and so on; from this relation

$$j_1 = \frac{\Sigma a_1a_2 - 3a_2a_3}{3a_1 - \Sigma a_r}, \text{ and so on.}$$

Substituting in (1), we obtain (left-hand number)

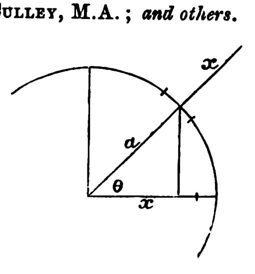
$$\frac{(3a_1-\Sigma a_r)}{(a_1^2-a_2a_3)} + \frac{(3a_2-\Sigma a_r)}{(a_2^2-a_3a_1)} + \frac{(3a_3-\Sigma a_r)}{(a_3^2-a_1a_2)}.$$

Since $\Sigma a_r = 0$, $(1) = 3\left\{\dfrac{a_1}{(a_1^2-a_2a_3)} + \dfrac{a_2}{(a_2^2-a_3a_1)} + \dfrac{a_3}{(a_3^2-a_1a_2)}\right\}$,

which reduces to zero; therefore $a \equiv$ focus of maximum inscribed ellipse in triangle m_r. The relation of a and j to the triad $\frac{1}{2}(a_r+j_r)$ is mutual; so that the same property for j may immediately be deduced.

12022. (Professor ZERR.) — A distance equal to the abscissa of a point is measured off on a normal to a circle; show that the locus of this point is $\rho^2 = 2a^2 - a^2\sin^2\theta \pm 2a^2\cos\theta$, + or - as the distance is measured outwards or inwards; prove that the area of this curve is $\frac{3}{2}\pi a^2 \pm 4a^2$.

Solution by J. M. STOOPS, B.A.; Prof. CULLEY, M.A.; *and others.*



$$\rho = a \pm x = a \pm a\cos\theta;$$

therefore $\rho^2 = a^2 + a^2\cos^2\theta \pm 2a^2\cos\theta$

$$= 2a^2 - a^2\sin^2\theta \pm 2a^2\cos\theta.$$

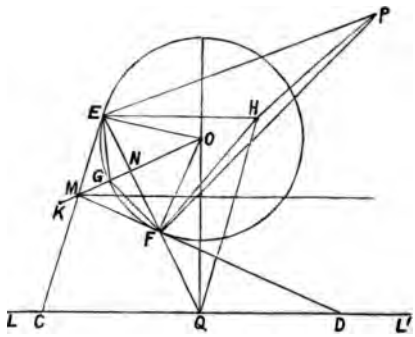
$$\text{Area} = 2\int_0^{\frac{1}{2}\pi}(2a^2 - a^2\sin^2\theta \pm 2a^2\cos\theta)\,d\theta$$

$$= 2\left[2a^2\theta - \frac{a^2\theta}{2} + \frac{a^2\sin 2\theta}{4} \pm 2a^2\sin\theta\right]_0^{\frac{1}{2}\pi}$$

$$= \tfrac{3}{2}\pi a^2 \pm 4a^2.$$

12029. (Professor Bourrienne.)—On donne, dans le plan d'un cercle O, une droite LL′ et un point P. Sur LL′, on considère deux points variables C, D, équidistants du centre O et extérieurs au cercle; des points C, D on mène deux droites tangentes en E, F au cercle et se coupant en M. Trouver (1) le lieu de M (deux cas); (2) les lieux des centres des cercles inscrit et circonscrit au triangle MEF, ainsi que le lieu du point de concours des hauteurs; (3) le lieu du centre du cercle circonscrit au triangle PEF.

Solution by J. M. Stoops, B.A.; Professor Bhattacharya; *and others.*



1. When the tangents from C, D are drawn symmetrically the solution is evident.

Case 2. Q is the foot of the perpendicular from O on LL′. MO meets the circle in G, and EF in N. NK is made equal to NO. H, the centre of the circle round EPF, is joined to E, F, P, and Q.

Consider $\triangle CDM$; its sides are cut in E, F, Q, hence

$$EM . FD . QC = EC . MF . QD.$$

For $QC = QD$, since C, D are equidistant from the centre; and $FD = EC$ for the same reason. Therefore EFQ is a straight line; therefore Q lies on the polar of M; therefore M lies on the polar of Q; *i.e.*, the locus of M is a fixed straight line.

2. In $\triangle MEF$, $\qquad \angle MEG = EFG = GEF$;

therefore G is the centre of the inscribed circle; that is, the locus of the centre of the circle inscribed in $\triangle MEF$ is the given circle itself.

OEMF is cyclic on account of the right angles; therefore the centre of the circumscribed circle of $\triangle MEF$ is the middle point of OM, but M travels along a fixed straight line; therefore the required locus is a straight line parallel to the locus of M, lying midway between that locus and the centre of the given circle.

In $\triangle MEF$, MN is drawn perpendicular to the opposite side, and produced to meet the circumscribed circle again in O; therefore the orthocentre lies at K. But the locus of N is a circle passing through the origin, namely the inverse of the locus of M. Hence, since $ON = NK$, the locus of K is a fixed circle.

3. HEF is an isosceles $\triangle$, and Q a point in the base produced; therefore $\qquad QH^2 - HE^2 = QE . QF = \text{constant}$;

therefore $\qquad QH^2 - HP^2 = \text{constant}$;

therefore the locus of H is a fixed straight line at right angles to PQ.

12150. (Dr. DONALD MACALISTER, M.A.)—A convex inextensible pliable envelope, in the form of a surface of revolution with its axis vertical, is exposed to water-pressure from within. Prove that at the widest part the tension along the meridians is a maximum or minimum according as it is less or greater than the tension across them.

Solution by H. W. CURJEL, B.A.; Prof. CLAYTON; *and others.*

Let σ = the density of the liquid, and ρ = the radius of curvature at the point P of the generating curve. Then, using the notation of Art. 126 of BESANT's *Hydromechanics* (fourth edition), we have

$$g\sigma(c-x) = p = \frac{t}{\rho} + \frac{t'\cos\theta}{y};$$

$$2\pi yt\cos\theta = \int_0^x g\sigma\pi y'\cdot dx' + g\sigma\pi y^2(c-x).$$

Differentiating the latter with respect to s,

$$2\frac{dy}{ds}t\cos\theta + 2y\frac{dt}{ds}\cos\theta - 2yt\sin\theta\frac{d\theta}{ds} = g\sigma y^2\frac{dx}{ds} + 2g\sigma y\frac{dy}{ds}(c-x) - g\sigma y^2\frac{dx}{ds};$$

$$\therefore\quad \cos\theta\frac{dt}{ds} = \left\{g\sigma(c-x) - \frac{t\cos\theta}{y} - \frac{t}{\rho}\right\}\frac{dy}{ds},\ \text{ for } \frac{d\theta}{ds} = -\frac{1}{\rho};$$

therefore, when $dy/ds = 0$, *i.e.* at the widest part, t has a critical value, which is a maximum or minimum as $g\sigma(c-x) - \dfrac{t\cos\theta}{y} - \dfrac{t}{\rho}$ is positive or negative, *i.e.* according as t is less or greater than t'.

[The PROPOSER's solution is as follows:—

Take the origin at the vertex, Ox along the axis, Oy along the tangent at the vertex. Then, if t is the tension along the meridians, t' the tension across the meridians, ds an element of arc of the generating curve at the point (x, y), it can be shown at once that

$$t' = \frac{d}{dy}(ty),\quad i.e.,\quad t' = \frac{d}{ds}(ty)\bigg/\frac{dy}{ds} = \left(y\frac{dt}{ds} + t\frac{dy}{ds}\right)\bigg/\frac{dy}{ds} = y\frac{dt}{ds}\bigg/\frac{dy}{ds} + t;$$

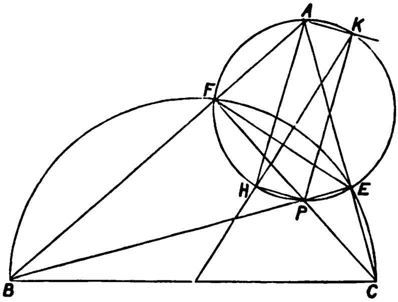
$$\therefore\quad \frac{t'-t}{y} = \frac{dt}{ds}\bigg/\frac{dy}{ds}.$$

But at the widest part $dy/ds = 0$, and as $(t'-t)/y$ is not infinite there, we must have also $dt/ds = 0$. The fraction on the right then assumes the indeterminate form 0/0. Its value is here $\dfrac{d^2t}{ds^2}\bigg/\dfrac{d^2y}{ds^2}$; and as the generating curve is concave to the axis of x, d^2y/ds^2 is negative. Thus d^2t/ds^2 is negative or positive according as t is $<$ or $>$ t'. In other words, at the widest part $dy/ds = 0$, and $dt/ds = 0$, and d^2t/ds^2 is $-$ or $+$ according as t is $<$ or $>$ t'.]

12157. (Professor DROZ-FARNY.)—Si de l'orthocentre d'un triangle on abaisse des perpendiculaires sur les bissectrices intérieure et extérieure

de l'angle A, la droite qui joint leurs pieds passe par le point milieu du côté BC.

Solution by W. J. Dobbs, B.A.; R. H. Whapham, M.A.; *and others.*



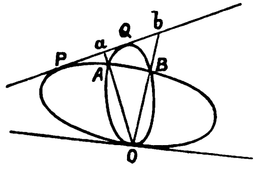
If P is the orthocentre, PH and PK perpendiculars on the internal and external bisectors of the angle A, a circle will go round FAKEPH, and the circle on BC as diameter passes through F, E.

The diameter HK of the former circle, since t bisects the arc FHE, must bisect the chord FE at right angles; and therefore passes through the centre of the second circle, that is through the mid-point of BC.

[*Otherwise*:—Since the mid-points of the three diagonals of a complete quadrilateral are collinear, the mid-points of AP, FE, and BC are collinear; hence HK bisects BC. Also HK is parallel to the join of A with the circumcentre. The Proposer remarks that "si dans le triangle ABC on construit les trois droites analogues à KH, elles se croisent au centre du cercle de Feuerbach. Si AB et AC sont les directions des diamètres conjugués égaux d'un système de coniques semblables, le point milieu de BC est le centre de l'hyperbole d'Apollonius correspondant au point P."]

12021. (Professor Clayton.)—Two conics touch at O. If P, Q be the points of contact of a common tangent, and A, B be their points of intersection (distinct from O), then the lines OP, OQ divide AB harmonically.

Solution by J. F. Hudson; H. W. Curjel, B.A.; *and others.*



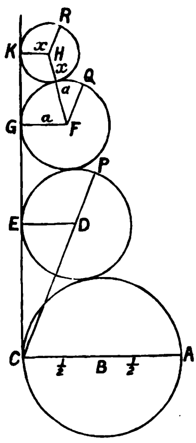
Let OA, OB meet PQ in *a*, *b* respectively. We may consider the two given conics and the lines OA, OB as three four-point conics, each going through the four points A, B, O, O, and PQ as a transversal. Then, by Desargue's theorem, *a* and *b* are a conjugate pair of points in the involution range whose double points are P, Q.

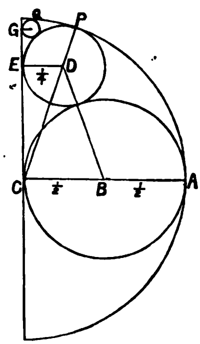
Hence $\qquad (PaQb) = -1, \quad$ or $\quad O(PAQB) = -1;$
therefore OP, OQ divide AB harmonically.

[*Otherwise*:—Project the line AB to infinity, and one of the conics into a circle; then the other also becomes a circle, and we get "If two circles touch, a common tangent subtends a right angle at the point of contact," which is evident from elementary geometry.]

12178. (R. Chartres.)—If a row of diminishing circles, each touching the preceding, be inscribed in a semi-circle, the first of the series being the maximum that can be inscribed, find the relation between the radii, and show that the fourth circle is $\frac{1}{10000}$ of the original circle, and that the radius of the twelfth circle is 1/131836324 of the radius of the semicircle.

Solution by Major-General O'Connell; the Proposer; *and others*.





Let
$AC = 1 = CDP = CFQ = CHR$, &c.
Also $HK = x$, $FG = a$.
From the two values of GK (Euc. III. 36, and I. 47),

$$(1-2x)^{\frac{1}{2}} - (1-2a)^{\frac{1}{2}} = (4ax)^{\frac{1}{2}},$$

or
$$x^2(1+2a)^2 - 2ax(3-2a) + a^2 = 0;$$

therefore product of the roots

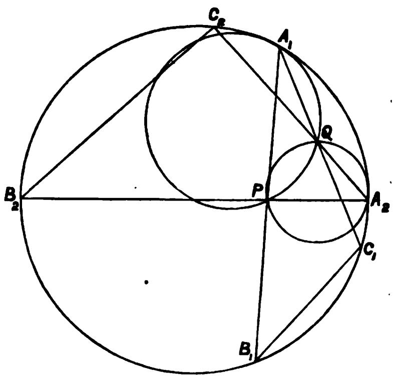
$$(HK \,.\, DE) = \frac{a^2}{(1+2a)^2},$$

or
$$\left(\frac{1}{HK} \cdot \frac{1}{DE}\right) = 2 + \frac{1}{a}.$$

Hence, if $1/r_1$, $1/r_2$, $1/r_3$ be the radii of three consecutive circles, $r_2 + 2 = (r_1 r_3)^{\frac{1}{2}}$, and the values can be found at once. Thus the reciprocals of the radii of the inscribed circles are 2, 4, 18, 100, $2\,(17)^2$, &c., the twelfth being $(11482)^2$.

530. (Editor.)—In a given circle inscribe a triangle so that two sides may pass through given points and that the third side may be a maximum or a minimum.

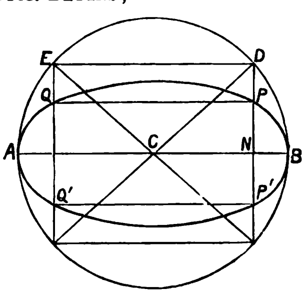
Solution by Professor Lampe.



Let P, Q be the given points, $A_1B_1C_1$ a triangle inscribed. If B_1C_1 is maximum or minimum, the opposite angle A_1 will be maximum or minimum too. The greatest angle, standing upon PQ, belongs to the least circumference passing through P and Q. Whence describe those circles which pass through P and Q, and touch the given circle. From the problem of taction two circles of this kind are known to exist: one touching in A_1, the other in A_2. In our figure there are two maxima: A_1 is the vertex on one side, A_2 on the other side of PQ, giving the greatest angle; B_1C_1 and B_2C_2 are the corresponding greatest sides.

12160. (Professor Zerr.)—Let AB be the transverse axis of an ellipse, C its centre; describe about the ellipse its auxiliary circle; draw the radii CD, CE of this circle, making angles of 45° with CB, CA respectively. Draw DPP′ perpendicular to AB, cutting the ellipse in P, P′; EQQ′ perpendicular to AB, cutting the ellipse in Q, Q′. Then will PP′Q′Q be the maximum rectangle inscribed in the ellipse.

Solution by Gertrude Poole, B.A.; Prof. Beyens; *and others.*



Let P be a point $(a\cos\alpha,\ b\sin\alpha)$ on the ellipse

$$x^2/a^2+y^2/b^2=1;$$

then rectangle

$$QPP'Q' = 4PN.CN$$
$$= 4ab\cos\alpha\sin\alpha$$

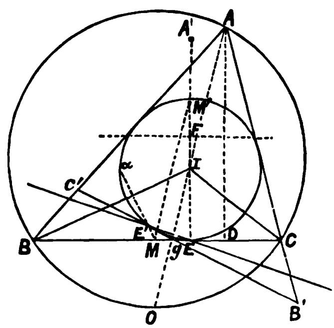
at a maximum $4ab(\cos^2\alpha-\sin^2\alpha)=0$ (differentiating); therefore

$$\cos 2\alpha = 0, \quad \therefore \quad \alpha = \tfrac{1}{4}\pi;$$

and this is obviously the maximum value, wherefore, &c.

12011. (J. H. GRACE.)—If ABC be a triangle, and I its in-centre, the nine-point circle of BIC passes through the point in which the in-circle of ABC is touched by the nine-point circle. Also, the orthocentre of BIC is the pole with respect to the in-circle of ABC of the line joining the mid-points of AB and AC.

Solution by Professor DROZ-FARNY ; R. KNOWLES, B.A. ; *and others.*



Soient E le point de contact du cercle inscrit avec BC, D le pied de la perpendiculaire abaissée de A sur BC. Portons sur AB, AC′ = AC et sur AC, AB′ = AB. La droite C′B′ sera tangente en E′ au cercle inscrit et coupera BC en g pied de la bissectrice AI. M étant le point milieu de BC et α le point d'intersection de ME′ avec le cercle inscrit, on a : $\mathrm{MD}\,.\,\mathrm{MG} = \mathrm{ME}'.\,\mathrm{M}\alpha$

$$= \left\{\tfrac{1}{2}(\mathrm{AB}-\mathrm{AC})\right\}^2 = (\mathrm{ME})^2.$$

On sait que α est le point de contact du cercle inscrit avec le cercle de Feuerbach du triangle primitif.

La bissectrice AI coupe le cercle circonscrit ABC en O et ce point est le centre du cercle circonscrit au triangle BIC. La parallèle MM′ menée par M au rayon OI jusqu'à sa rencontre avec la hauteur IE est le diamètre du cercle des neuf points du triangle BIC. Or EE′ étant perpendiculaire sur GI est aussi perpendiculaire sur MM′ et par conséquent l'inverse de ce cercle des neuf points pour M comme pôle et ME comme rayon d'inversion n'est rien d'autre que la droite EE′. Mais comme $(\mathrm{ME}^2) = \mathrm{ME}'.\,\mathrm{M}\alpha$, α est un point de ce cercle.

Soient A′ l'orthocentre du triangle BIC, et F le point d'intersection de la hauteur avec la droite qui joint les points milieux de AB et AC, on a

$$\mathrm{IE}\,.\,\mathrm{EA}' = \mathrm{BE}\,.\,\mathrm{EC} = (p-b)(p-c) \text{ donc } \mathrm{EA}' = \frac{(p-b)(p-c)\,p}{\Delta};$$

$$\mathrm{IA}' = \frac{(p-b)(p-c)\,p}{\Delta} - \frac{\Delta}{p} = \frac{p^2(p-b)(p-c) - p(p-a)(p-b)(p-c)}{\Delta p}$$

$$= \frac{(p-b)(p-c)[a]}{\Delta};$$

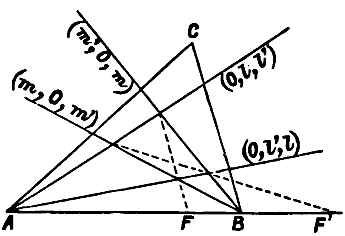
$$\mathrm{IF} = \frac{h}{2} - \frac{\Delta}{p} = \frac{\Delta}{a} - \frac{\Delta}{p} = \frac{\Delta(p-a)}{ap};$$

$$\mathrm{IA}'.\,\mathrm{IF} = \frac{\Delta(p-a)(p-b)(p-c)\,a}{ap\Delta} = \frac{p(p-a)(p-b)(p-c)}{p^2} = \frac{\Delta^2}{p^2} = r^2.$$

12040. (EDITOR.)—Through each angle of a triangle let two straight lines be drawn, equally inclined to the bisectors of those angles, but the

inclination not necessarily the same for each of the pairs; prove that the straight lines joining the intersections of these lines will meet the corresponding sides of the triangle in three points which will be in the same straight line.

Solution by GERTRUDE POOLE, B.A.; Prof. SARKAR; *and others.*



$l\beta - l'\gamma = 0$, $l'\beta - l\gamma = 0$ are two lines equally inclined to the bisector of A; and

$m\gamma - m'\alpha = 0$, $m'\gamma - m\alpha = 0$

two lines equally inclined to the bisector of B.

The point of intersection the first pair is given by

$\alpha/lm = \beta/l'm' = \gamma/lm'$,

and of the latter pair by

$\alpha/l'm' = \beta/lm = \gamma/l'm$.

Therefore the line joining them is given by the determinant

$$[\alpha, \beta, \gamma;\ lm,\ l'm',\ lm';\ l'm',\ lm,\ l'm] = 0;$$

therefore F is given by $\alpha/(ll'm'^2 - ll'm^2) + \beta/(l_1^2mm' - l'^2mm') = 0$;

hence we have $\qquad \dfrac{\text{AF}}{\text{FB}} = -\dfrac{\sin \text{B}}{\sin \text{A}} \cdot \dfrac{ll'}{mm'} \cdot \dfrac{m'^2 - m^2}{l'^2 - l^2}.$

For BD/DC and CE/EA, we shall get similar expressions with m, n; n, l respectively in place of l, m; whence, by multiplication, we shall get AF . BD . CE/FB . DC . EA $= -1$, so that D, E, F will be collinear.

[If we take the case of the point F′, whose nature is sufficiently indicated in the figure, we shall get AF′/F′B $= -$AF/FB, either by harmonic properties of the little quadrilateral in the middle, or by interchanging l and l' (or m and m', but not both) throughout the algebraical work; similarly, BD′/D′C $= -$ BD/DC, and CE′/E′A $= -$CE/EA. Whence D, E, F, D′, E′, F′ lie three by three on four straight lines, viz., DEF (already shown), DE′F′, D′EF′, D′E′F.]

12166. (J. C. MALET, F.R.S.) — Prove that (1) a plane curve of order 17 cannot have more than 85 cusps; (2) when it has this number of cusps it can have no other double points; and (3) no unicursal plane curve of order 17 can have more than 22 cusps.

Solution by Profs. SCHOUTE, KRISHNAMACHARRY, *and others.*

The formulas of PLUECKER are

$$\nu = 272 - 2\delta - 3x, \quad \iota = 765 - 6\delta - 8x, \quad \tau = \delta + \tfrac{1}{2}(\nu - 17)(\nu + 8).$$

We distinguish the cases $\nu = 17$, $\nu > 17$, $\nu < 17$. For $\nu = 17$, the

first formula gives $2\delta+3x = 255$, *i.e.* maximum value of $x = 85$. This case is possible ($\mu = \nu = 17$, $\delta = \tau = 0$, $x = i = 85$).

For $\nu > 17$, the first formula gives $2\delta+3x < 255$, *i.e.* maximum value of $x < 85$. For $\nu < 17$, we put $x = 85+k$, and find

$$\nu = 17-2\delta-3k,\quad i = 85-6\delta-8k,\quad \tau = \delta+\tfrac{1}{2}(2\delta+3k)(2\delta+3k-25).$$

The last equation shows that τ will be zero or positive under each of the two conditions $\qquad 2\delta+3k \leqq \frac{25}{2}-\frac{1}{2}(625-8\delta)^{\frac{1}{2}}$,

$$2\delta+3k \geqq \tfrac{25}{2}+\tfrac{1}{2}(625-8\delta)^{\frac{1}{2}}.$$

When the first is fulfilled, we have, *a fortiori*,

$$2\delta+3k = \tfrac{25}{2}-\tfrac{1}{2}(625-68)^{\frac{1}{2}}, \quad \text{since} \quad 2\delta+3k<17.$$

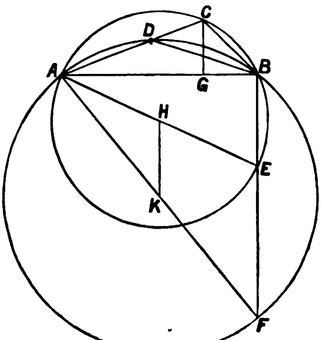
This demands $2\delta+3k < 1$, or $\delta = k = 0$. And in the second case we have, *a fortiori*, $2\delta+3k = \frac{25}{2}+\frac{1}{2}(625-68)^{\frac{1}{2}}$, or $2\delta+3k \geqq 25$, which is inconsistent with the condition $2\delta+3k<17$. The only result that can be admitted is, therefore, $\delta = 0$, $k = 0$.

If the curve be unicursal, we have $\delta+x = 120$. Then the second formula can be reduced to $i = 45-2x$, which proves that the maximum value of x is 22. This case is possible ($\mu = 17$, $\nu = 10$, $\delta = 98$, $\tau = 35$, $x = 22$, $i = 1$).

[For $\mu = 3p+2$, we also find the maximum value of $x = p(3p+2)$ for $\mu = \nu$, &c.]

12073. (M. Brierley.)—ABC is a plane triangle, obtuse at C, inscribed in the circle ACBE; perpendicular to the base AB, draw BE meeting the circle in E, and produce it so that EF = AC. Through the points A, B, F, describe another circle intercepting AC in D. Then will the distance of the centres of the two circles be equal to $\frac{1}{2}$AC, and CD will be equal to the perpendicular from C upon the base AB.

Solution by W. J. Dobbs, B.A.; Prof. Mukhopadhyay; *and others.*



Take H, K the centres of the circles, so that H, K are respectively the mid-points of AE, AF; then we have

$$HK = \tfrac{1}{2}EF = \tfrac{1}{2}AC.$$

Now $\qquad \angle CDB = AFE$,

and $\qquad \angle BCD = AEF$;

therefore the triangles BCD, AEF are similar; therefore

$$EF \cdot CB = CD \cdot AE,$$

that is $AC \cdot CB = CD \cdot AE$.

But, by a well-known theorem,

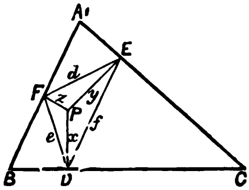
$$AC \cdot CB = CG \cdot AE,$$

when CG is perpendicular on AB;

therefore $\quad CD = CG$.

12170. (J. Griffiths, M.A.)—(1) If the sides d, e, f and area Δ of the pedal triangle DEF of a point P with reference to a given triangle ABC, satisfy a relation $ld^2+me^2+nf^2=k\Delta$, where l, m, n, k are constants, prove that the locus of P will be a pair of circles inverse to each other with respect to the circumcircle ABC. For example, if ω be the Brocard angle of ABC, and $d^2+e^2+f^2=4\Delta\cot\omega$, the locus of P will be the Brocard circle of ABC, and its inverse, viz., the Lemoine line. (2) If the sides d, e, f are to each other in the duplicate ratio of the corresponding sides of the given (acute-angled) triangle ABC, *i.e.*, if $d/a^2=e/b^2=f/c^2$, prove that P will be one or other of two given inverse points on the line joining the centre of the circumcircle and orthocentre of ABC. In this case the distances x, y, z of P from A, B, C are proportional to the opposite sides a, b, c; *i.e.*, $x/a=y/b=z/c$. [The pedal triangles of two inverse points are similar to each other.]

Solution by Professor Schoute.



If x, y, z be the trilinear coordinates DP, EP, FP of P, we find

$$d^2=y^2+z^2+2yz\cos A,$$
$$e^2=z^2+x^2+2zx\cos B,$$
$$f^2=x^2+y^2+2xy\cos C,$$
$$2\Delta=yz\sin A+zx\sin B+xy\sin C.$$

Also,

$$d^2=0,\quad e^2=0,\quad f^2=0,\quad \Delta=0$$

are the equations of the three point-circles A, B, C, and of the circumcircle ABC.

Therefore $$ld^2+me^2+nf^2=\pm k\Delta \quad\text{.............................}(1)$$

represents two circles. And by the solution of Question 12113 it is immediately seen that these two circles are inverse to each other with respect to the circumcircle ABC. For we have

$$\frac{d^2}{d'^2}=\frac{e^2}{e'^2}=\frac{f^2}{f'^2}=\frac{\Delta}{-\Delta'},$$

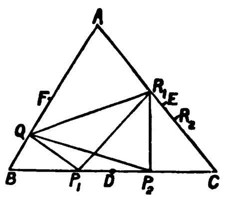
where (d, e, f, Δ) and (d', e', f', Δ') belong to the points P and P′ of the diagram, used in the solution of that Question.

For $l=m=n=1$, $k=4\cot\omega$, the equation (1) can be thrown into the form $\cot\omega=\pm\dfrac{d^2+e^2+f^2}{4\Delta}$, which shows that the Brocard-angles of the triangles ABC and DEF are equal. In that case, the equation (1) represents the locus of the point P, the pedal triangle DEF of which corresponds in Brocard angle with ABC. For the + sign the triangles DEF and ABC are isotrope (locus = Brocard circle), for the − sign the triangles DEF and ABC are anisotrope (locus = line of Lemoine).

[The circle $ld^2+me^2+nf^2=\Delta$ can be represented by the point with the coordinates l, m, n. For the study of this representation, compare the *Sitzungsberichte* of Vienna, Vol. XCIV., page 786.]

1211. (S. WATSON.)—Show that the *average* area of all the triangles that can be inscribed within a given triangle, is one-fourth of the triangle.

Solution by R. CHARTRES; Professor BHATTACHARYA; *and others.*



Let D, E, F be the mid-points of the sides, and $DP_1 = DP_2$; then, since

$$\Delta QR_1P_1 + \Delta QR_1P_2 = 2\Delta QR_1D,$$

therefore ΔQR_1D is the average of all the inscribed triangles standing on QR_1; but the average of all the inscribed triangles standing on QD is similarly

$$= \Delta QDE = \Delta FDE = \tfrac{1}{4}ABC.$$

11989. (Professor ZERR.)—A smooth, straight, thin tube, length 2, is balancing horizontally about its middle point, and a particle is shot into it horizontally with such a given velocity that it just arrives at the middle point of the tube. If the angular velocity communicated to the tube equals the given velocity of the particle, show that the weight of the particle is one-third the weight of the tube.

Solution by H. W. CURJEL, B.A.; Prof. AIYAR; *and others.*

Since the potential energy of the system is the same when the particle is at the middle of the tube as at the beginning of the motion, the kinetic energy is the same. Therefore, $\frac{1}{2}mv^2 = \frac{1}{2} \times \frac{1}{3}M\omega^2$, where m, M are the masses of the particle and the tube, and v the velocity of the particle, and ω the angular velocity of the tube; $\therefore$ if $v = \omega$, $m = \frac{1}{3}M$.

10399, 12172. (F. R. J. HERVEY.) — Of the three quadrilaterals formed by four lines, let X, Y denote any two, each with a definite sense of description; and let p, q, r, s be the ratios of the sides of Y, taken in order, to the corresponding (that is, collinear) sides of X. Prove that, if we start with that line on which the sides of X and Y are represented with their proper sense by LM, MN respectively, we have

$$(q-p-1)(r-s-1) = 1, \quad qr = ps.$$

Solution by the PROPOSER.

Let LL′, MM′, NN′ be the three diagonals; any case of X, Y may be represented by LML′M′, MNM′N′, in which, since either L, M, N or

L′, M′, N′ must be collinear, we may assume the former. Let $\alpha, \beta, \gamma, \delta$ be the vector sides of X, named in the order of Y; we have:—

$$\begin{aligned} LM + MN + NL &= 0, \\ -L'M' + NM' + L'N &= 0, \\ M'L - M'N' + LN' &= 0, \\ ML' + N'M - N'L' &= 0; \end{aligned}$$

$$\alpha + \quad \beta + \quad \gamma + \quad \delta = 0 \;\;\ldots\ldots\ldots\; (X),$$
$$p\alpha + \quad q\beta + \quad r\gamma + \quad s\delta = 0 \;\;\ldots\ldots\ldots\; (Y),$$
$$(-1-p)\alpha + (1-q)\beta + (-1+r)\gamma + (1+s)\delta = 0.$$

In the first set, the terms of the second (or Y) column read cyclically in order; the other columns cyclically, though not in order. The terms of the equations of the second set correspond exactly to the columns of the first without their prefixed signs, the coefficients of the last equation being thereby determined.

From the second set we may eliminate any two vectors and equate to zero the coefficients of the others, determinants formed by taking the columns of the coefficients in threes; we have thus the four equations

$$q(1-r) = s(1-q), \qquad p(1+s) = r(1+p),$$
$$s(1+p) = q(1+s), \qquad r(1-q) = p(1-r).$$

Each of these, in fact, expresses the well-known property of the segments of the sides of a triangle made by a transversal, and any two of them will be found to be equivalent to the proposed relations (excepting that the latter are satisfied by $p = q$, $r = s$).

With a given figure, there are 24 ways of assigning the meaning of X, Y; it is evident that these correspond respectively to the 24 ways of writing L, M, N, collinear on the figure.

12140. (J. Griffiths, M.A.)—If the sides of the pedal triangle DEF of a point P, with reference to a given triangle ABC, be connected by a homogeneous and integral algebraic relation of the form $\phi(d, e, f) = 0$, prove that the equation of the locus of P will be $\phi(ax, by, cz) = 0$, where x, y, z are the distances of P from the vertices A, B, C of the given triangle whose sides are a, b, c. [The curve represented by $\phi(ax, by, cz) = 0$ will be of an even degree and self-inverse with respect to the circumcircle ABC. For example, if $\phi(d, e, f) \equiv (u, v, w, u', v', w')(d^2, e^2, f^2)^2$, then the locus of P will be a bicircular quartic, self-inverse with regard to the circumcircle ABC. This includes the case of the pedal triangle DEF having a constant Brocard angle.]

Solution by Professors Droz-Farny, Ignacio Beyens, *and others.*

Dans le quadrilatère inscriptible PEAF on a évidemment:

$$EF = PA \sin A, \quad \text{ou} \quad d = x \sin A, \quad d = ax/2R;$$

si donc la relation $\phi(d, e, f)$ est homogène elle se transforme immédiatement après suppression du facteur d'homogénéité en $\phi(ax, by, cz) = 0$.

Pour deux points inverses par rapport à la circonférence ABC on a (voir Quest. 12113) $x : x' = y : y' = z : z'$; donc

$$ax : ax' = by : by' = cz : cz'.$$

Si donc la relation $\phi(ax, by, cz) = 0$ est homogène en ax, &c., on aura aussi $\phi(ax', by', cz') = 0$, d'où la remarque de Mr. GRIFFITHS.

Pour les courbes du quatrième degré en coordonnées tripolaires, voir le bel article de M. A. POULAIN dans le *Journal de Mathématiques*.

12180. (A. E. THOMAS, M.A.)—Find a general value for x making $\square \equiv (x+a^2)(x+b^2)(x+c^2)$ a perfect square.

Solution by D. BIDDLE; Prof. BHATTACHARYA; *and others.*

Let $\square = (abc+n)^2$; then

$$x^3 + (a^2+b^2+c^2)x^2 + (a^2b^2+a^2c^2+b^2c^2)x - (n^2+2n\,.\,abc) = 0.$$

By taking $x = z - \frac{1}{3}(a^2+b^2+c^2)$, we remove the second term, and obtain $z^3+qz+r=0$, in which $q = \frac{1}{3}(a^2b^2+a^2c^2+b^2c^2-a^4-b^4-c^4)$,

$r = -\frac{1}{27}\left[\left\{5(a^2b^2+a^2c^2+b^2c^2)-2(a^4+b^4+c^4)\right\}(a^2+b^2+c^2)+27n^2+54nabc\right]$.

Moreover, by CARDAN's method, we have

$$x = \left\{-\tfrac{1}{2}r + (\tfrac{1}{4}r^2+\tfrac{1}{27}q^3)^{\frac{1}{2}}\right\}^{\frac{1}{3}} + \left\{-\tfrac{1}{2}r - (\tfrac{1}{4}r^2+\tfrac{1}{27}q^3)^{\frac{1}{2}}\right\}^{\frac{1}{3}} - \tfrac{1}{3}(a^2+b^2+c^2),$$

which our data readily enable us to reduce.

Thus, let $a=2$, $b=3$, $c=4$; then $abc=24$, $a^2b^2+a^2c^2+b^2c^2=244$, $a^2+b^2+c^2=29$, and $a^4+b^4+c^4=353$, making $q=-36\cdot\dot{3}$, and

$$r = -(552\cdot\dot{0}7\dot{4}+48n+n^2).$$

Let $n=1$, then $q=-36\cdot\dot{3}$ as before, and $r=-601\cdot\dot{0}7\dot{4}$; whence

$x = \left\{300\cdot5\dot{3}7\dot{0}+297\cdot567\right\}^{\frac{1}{3}} + \left\{300\cdot5\dot{3}7\dot{0}-297\cdot567\right\}^{\frac{1}{3}} - 9\cdot\dot{6} = 0\cdot1962$ nearly; and we find that $(4\cdot1962)(9\cdot1962)(16\cdot1962) = 625 = 25^2$, also very nearly. That this result is merely approximate is due entirely to our inability to extract the square and cube roots with numerical accuracy. The value of x as at first stated is absolutely correct.

11225. (Professor SHIELDS, M.A.)—In a rectangular tract of land whose length is to its width as 60 to 12, a diagonal line from the lower south-west corner A to the upper diagonal corner B, strikes a stake S, whose perpendicular distance from the line AC on the south side of the land is 200 yards, and the diagonal distance from the corner at A to the stake S is 40 yards greater than the width of the tract from B to C. Give the dimensions and area of the land.

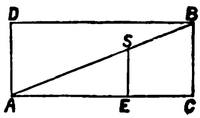
Solution by Professors ZERR, BEYENS, *and others.*

Let $AC = 60x'$, $BC = 12x'$; then

$SE = 200, \quad AS = 12x' + 40, \quad AB = 12x'\sqrt{26},$

From the triangles ASE, ABC,

$$12x' + 40 : 200 = 12x'\sqrt{26} : 12x'.$$



Therefore $x' = \frac{10}{3}(5\sqrt{26} - 1)$,

$$60x' = 200(5\sqrt{26} - 1) = \text{length}, \quad 12x' = 40(5\sqrt{26} - 1) = \text{width},$$

$$60x' \times 12x' = 8000(5\sqrt{26} - 1)^2 = \text{area}.$$

11494. (Professor CHAKRIVARTI.)—Find the angles of a triangle in which the greatest side is twice the least, and the greatest angle twice the mean angle. Prove that a triangle whose sides are as 17156 : 13395 : 8578 is a very approximate solution.

Solution by Professors ZERR, AIYAR, *and others.*

From the figure, $\cos A = b/a$,

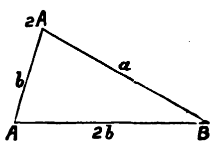
also $\quad \cos A = \sqrt{(a+b)^2 - 4b^2}/2\sqrt{ab}$;

$\therefore \quad a^4 + 2a^3b - 3a^2b^2 - 4ab^3 = 0,$

$\therefore \quad a = \frac{1}{2}b(\sqrt{17} - 1)$

or $\quad a : b = 13394{\cdot}8999 : 8578$

$\quad = 13395 : 8578$, nearly.



$$\cos A = 2/(\sqrt{17} - 1) = {\cdot}6403882, \quad A = 50^\circ\,10'\,45'', \quad B = 29^\circ\,27'\,45''.$$

11600. (I. ARNOLD.)—A given straight line AB is bisected in C, and CD is drawn perpendicular to AB. A point P is taken, and PA and PB joined, PA cutting the perpendicular CD in E. Find the locus of a point P, such that PE is always equal to PB.

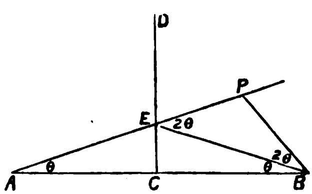
Solution by H. W. CURJEL, B.A.; Professor ZERR; *and others.*

Take CB, CD as axes of x and y, and let $\angle PAB = \theta$; then, if

$$CB = a, \quad \tan\theta = y/(a+x),$$

$$\cos^2\theta = (a+x)^2/\{y^2 + (a+x)^2\},$$

$$x = EP\cos\theta = \frac{1}{2}\,\frac{a}{\cos\theta\cos 2\theta}\cos\theta;$$



thus locus of P is

$$\frac{2\,(a+x)^2}{y^2 + (a+x)^2} = \frac{(a-2x)}{2x}, \quad \text{or} \quad y^2(a-2x) = (6x-a)(x+a)^2.$$

11752. (Professor ZERR.)—Trace the curve

$$y = \frac{8}{3\pi}\int_0^\infty \frac{\sin^3 x\theta \sin^2\theta}{\theta^5}\,d\theta.$$

Solution by the PROPOSER.

$$\frac{dy}{dx} = \frac{8}{\pi}\int_0^\infty \frac{\sin^2 x\theta \cos x\theta \sin^2\theta}{\theta^4}\,d\theta = \frac{2}{\pi}\int_0^\infty \frac{\cos x\theta \sin^2\theta - \cos 3x\theta \sin^2\theta}{\theta^4}\,d\theta;$$

$$\int_0^\infty \frac{\cos b\theta}{\theta^4}\,d\theta = \frac{\pi b^3}{12} \text{ if } b \text{ is positive}, = -\frac{\pi b^3}{12} \text{ if } b \text{ is negative}.$$

If $x > 2$,

$$dy/dx = \tfrac{1}{24}\left\{2x^3-(x+2)^3-(x-2)^3 - 2(3x)^3+(3x+2)^3+(3x-2)^3\right\} = 2x;$$

$$y = x^2, \text{ a parabola.}$$

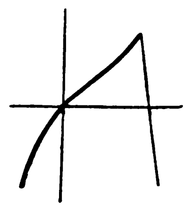
If $2 > 3x > 0$,

$$dy/dx = \tfrac{1}{24}\left\{2x^3-(x+2)^3+(x-2)^3-2(3x)^3 + (3x+2)^3-(3x-2)^3\right\}$$

$$= \tfrac{1}{6}(24x^2-13x^3);$$

$$y = \tfrac{1}{24}(32x^3-13x^4).$$

The curve is somewhat as figured.



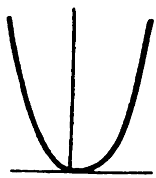
If $2 > x > 0$ and $< 3x$,

$$dy/dx = \tfrac{1}{24}\left\{2x^3-(x+2)^3+(x-2)^3-2(3x)^3 + (3x+2)^3+(3x-2)^3\right\}$$

$$= \tfrac{1}{12}(x^3-6x^2+36x-8);$$

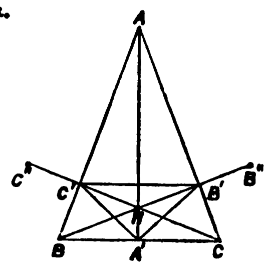
$$y = \tfrac{1}{48}(x^4-8x^3+72x^2-32x).$$

The curve is somewhat as figure.



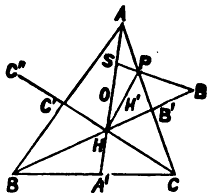
9729. (Rev. T. ROACH, M.A.)—Inscribe an ellipse in an isosceles triangle so that one focus may be at the orthocentre.

Solution by Prof. SCHOUTE *and the* PROPOSER.



Professor SCHOUTE states that an ellipse inscribed in an isosceles triangle, one focus of which is at the orthocentre, is immediately found.

When AH = HA′, B′B″ = HB′, C′C″ = HC′, the centre of the circle A″B″C″ is the second focus of the ellipse, for then the circle A″B″C″ evidently is the director-circle.



Mr. ROACH states that if H be the orthocentre, then the other focus must lie on the line AHA′.

For, if otherwise, as H′, then the angle HAB $=$ H′AB′; $\therefore$ HAB′ $=$ H′AB′;

$\therefore$ the other focus lies on AHA′ (TAYLOR, *Geometry of Conics*, Art. 98).

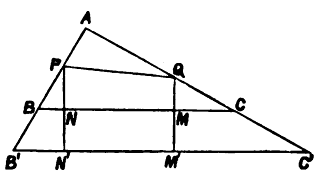
Make the angle CHP $=$ CHA, then P is point of contact (TAYLOR, Art. 39).

Produce B″P to meet AA′ in S, and S is the other focus.

Mr. ROACH adds that, by symmetry, the centre of A′B″C″ lies on AA′, as O; then OB″ $=$ OA′; but SB″ $=$ diameter of auxiliary circle, $\therefore$ SB″ $>$ SA, $\therefore$ the centre of the circle A′B″C″ is not the second focus.

12005. (Professor ROUJEAU.)—Par un point P pris sur le côté AB d'un triangle ABC mener une droite telle que la partie PQ comprise entre les côtés AB et AC soit égale à la somme des perpendiculaires abaissées des points P et Q sur BC.

Solution by H. J. WOODALL, A.R.C.S.; Prof. BHATTACHARYA; *and others*.



Draw PN perpendicular to BC, and produce to N′ making NN′ $=$ PN. Through N′ draw B′C′ parallel to BC. Let Q be the required position of the point. Draw QMM′ perpendicular to B′C′, join PQ. Then

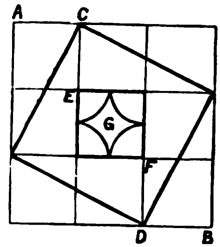
$$PQ = PN + QM = NN' + QM$$
$$= MM' + QM = QM',$$

i.e., the locus of Q is parabola, focus P, directrix B′C′.

11890. (Professor SHIELDS.)—(Connected with Quest. 11744.)—There is a large square piece of land AB, in which is laid off another less square piece CD, whose sides are oblique to the sides of the large square, and the four corners of the inner square CD touch the sides of the large square at a distance equal to $\frac{1}{3}$ of their length from the corners; the inner square CD, being divided by lines drawn from each of the four corners to the middle of the opposite sides, forms another square EF in the centre. These four lines and sides of the square EF are parallel to

the sides of the large square AB. From each corner of the centre square EF, as centres, are drawn four equal quadrants q, q, q, q tangential to each other, thus enclosing $2\frac{73}{500}$ acres of land in the centre G. Required the side and area of each square; also area of one quadrant.

Solution by T. SAVAGE, M.A.; Prof. ZERR; *and others.*



Obviously, AB = 9EF, CD = 5EF.

Denoting side of EF by s, we have

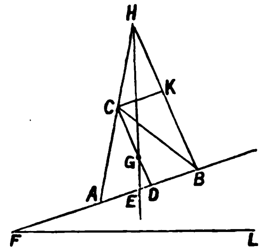
$$G = 2\tfrac{73}{500} \text{ acres} = s^2 - 4\,.\,(\tfrac{1}{2}s)^2.\tfrac{1}{4}\pi$$
$$= s^2\left\{\tfrac{1}{4}(4-\pi)\right\};$$
$$\therefore\ s^2 = 2\tfrac{73}{500} \times 4/(4-\pi).$$

Again, area of quadrant

$$= (\tfrac{1}{2}s)^2\tfrac{1}{4}\pi = 2\tfrac{73}{500} \times \frac{\pi}{4(4-\pi)}.$$

9550. (The EDITOR.)—An equilateral triangular lamina, having its plane vertical and its base in contact with a given inclined plane, is supported by a string fastened at its vertex; prove that the string can range, consistently with equilibrium, through an angle $\cot^{-1}\{2/\sqrt{3} - \cot\alpha\}$, where α is the inclination of the plane.

Solution by Profs. ZERR, KRISHNAMACHARRY, *and others.*



Let ABC be the equilateral triangle with its base AB on the inclined plane FB, G its centre of gravity. Through G draw GH vertical, meeting the perpendicular to the plane BH in H. Then the string can range between CH and CD. Now $GD = \frac{1}{6}a\sqrt{3}$, $\angle CBK = 30°$, $\angle BFL = \alpha$, a = side of triangle; then

$$DE = \tfrac{1}{6}a\sqrt{3}\tan\alpha,$$

$$BE = \frac{1}{2\sqrt{3}}(\tan\alpha + \sqrt{3}),$$

$$BH = \frac{1}{1\sqrt{3}}a + \tfrac{1}{2}a\cot\alpha,\quad CK = \tfrac{1}{2}a,\quad BK = \tfrac{1}{2}\sqrt{3}a;$$

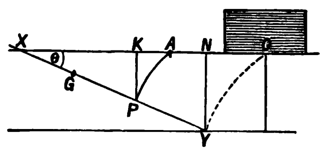
$$\therefore\ HK = \tfrac{1}{2}a\cot\alpha - \frac{a}{\sqrt{3}} \text{ and } \cot HCK = \frac{1}{\cot\alpha - 2/\sqrt{3}};$$

$$\therefore\ \cot HCD = \cot(\tfrac{1}{2}\pi + HCK) = 2/\sqrt{3} - \cot\alpha;$$

$$\therefore\ HCD = \cot^{-1}\{2/\sqrt{3} - \cot\alpha\}.$$

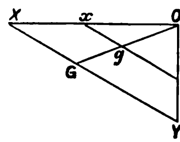
12167. (W. J. Dobbs, B.A.)—A box containing a large number of small shot is compelled to move in vacuo along a fixed horizontal straight line with uniform acceleration f, starting from a fixed point O. As it moves the shot are continually falling out of the box through a small hole. Show that at any instant the shot which have fallen out of the box are arranged along a straight line which is in a fixed direction. Prove also that the centre of mass of the trail of shot divides it in the ratio 1 : 2, and that the path of the centre of mass is a straight line along which it moves with uniform acceleration.

Solution by C. Morgan, M.A., R.N.; Prof. Shields; *and others.*



1. Let t, τ be times from O to X, A; X, P, Y simultaneous positions of shot; then

$$\frac{XK}{PK} = \frac{\frac{1}{2}f(t-\tau)^2}{\frac{1}{2}g(t-\tau)^2} = \frac{f}{g} = \frac{\frac{1}{2}ft^2}{\frac{1}{2}gt^2} = XN/NY;$$



hence P lies in XY, inclined at $\tan^{-1} g/f$ to the horizon.

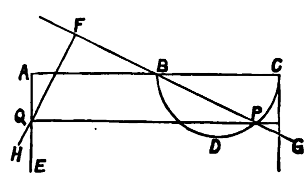
2. $XP = \frac{1}{2}f(t-\tau)^2 . \sec\theta$. If G is the centre of mass of the trail of shot,

$$XG = \frac{1}{n}\frac{1}{2}f\left\{\left(\frac{n}{n}\right)^2 t^2 + \left(\frac{n-1}{n}\right)^2 t^2 + \ldots + \left(\frac{1}{n}\right)^2 t^2\right\}\sec\theta,$$

where τ is put successively $0/n$, t/n, $2t/n \ldots$,

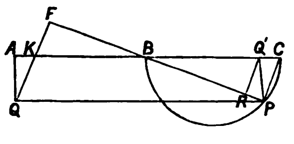
$$= \tfrac{1}{2}ft^2\,\frac{n(n+1)(2n+1)}{6n^3}\sec\theta = \tfrac{1}{2}ft^2 = \tfrac{1}{3}\sec\theta; \text{ in the limit } = \tfrac{1}{3}XY.$$

3. If the initial velocity is zero, then XG : GY = 1 : 2, and XY is fixed in direction; therefore the centre of mass moves along straight line OG with uniform acceleration.



12204. (D. Biddle.)—Let the straight line (= unity), AB, be produced to C, so that BC = AB, and on BC describe the semicircle BDC; also draw AE at right angles to AC, and let BF (< 1) be a value of which it is required to find the cube-root. Prove that, if FH be drawn at right angles to FG, and the system of lines GF, FH be supposed to revolve about B, until the semicircle and the perpendicular (AE) be cut by FG, FH respectively in P and Q, points equidistant from AC, then BP is the cube-root of BF.

Solution by R. H. W. WHAPHAM; GERTRUDE POOLE, B.A.; *and others.*



Draw PQ′ perpendicular to BC, and Q′R perpendicular to BP. Then AK evidently = Q′C; therefore

$$BK = BQ', \quad \therefore\ BF = BR.$$

But $BR : BQ' = BQ' : BP = BP : 1$;

therefore $BP^3 = BR = BF$;

therefore $BP = \sqrt[3]{BF}$.

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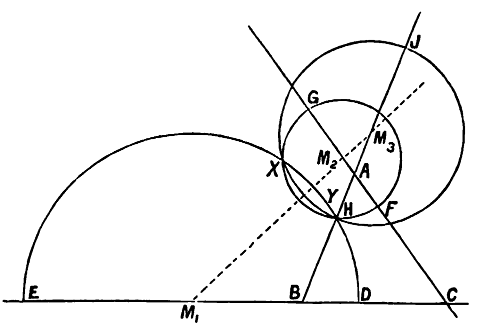
862. (EDITOR.)—Given three points, find a fourth point, in the same plane, such that its distance from each of the given points may have one to another given ratios.

Solution by Profs. LAMPE, BHATTACHARYA, *and others.*

(1) *Geometrical construction.*—Let A, B, C be the given points, X the fourth point; $XA : XB : XC = \alpha : \beta : \gamma$. Divide BC internally and



externally in the ratio $\beta : \gamma$, *i.e.*, $DB : DC = EB : EC = \beta : \gamma$, bisect DE in M_1, and draw a circle about M_1 as centre, with M_1D as radius.

This circle is a locus for X. There are still two analogous circles on which X is situated, M_2 and M_3, M_1, M_2, M_3 lying in one line. These three circles intersect in two points X and Y (which may be imaginary).

2. *Trigonometrical calculation.*—Let $BC = a$, $CA = b$, $AB = c$, $XA = d$, $XB = e$, $XC = f$. Then $d : e : f = \alpha : \beta : \gamma$, or $d = \alpha\kappa$, $e = \beta\kappa$, $f = \gamma\kappa$, κ being a numerical factor.

Six lines a, b, c, d, e, f joining four points in a plane are connected by the relation

$$a^2(d^2-e^2)(d^2-f^2)+b^2(e^2-f^2)(e^2-d^2)+c^2(f^2-d^2)(f^2-e^2)$$
$$+a^2d^2(a^2-b^2-c^2)+b^2e^2(b^2-c^2-a^2)+c^2f^2(c^2-a^2-b^2)+a^2b^2c^2=0$$

(see Salmon, *Geometry of Three Dimensions*, § 50). Substituting the above values for d, e, f, this relation becomes

$$\kappa^4 \left\{a^2(\alpha^2-\beta^2)(\alpha^2-\gamma^2)+b^2(\beta^2-\gamma^2)(\beta^2-\alpha^2)+c^2(\gamma^2-\alpha^2)(\gamma^2-\beta^2)\right\}$$
$$+\kappa^2\left\{a^2\alpha^2(a^2-b^2-c^2)+b^2\beta^2(b^2-a^2-c^2)+c^2\gamma^2(c^2-a^2-\beta^2)\right\}+a^2b^2c^2=0.$$

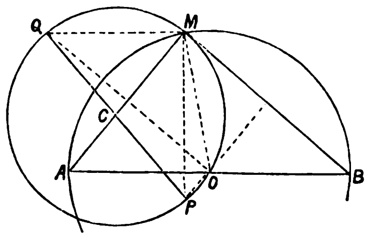
Solving this quadratic equation for κ^2, we obtain two values for κ, and henceforth two sets of values for d, e, f.

Example: $a = 6$, $b = 7{\cdot}2$, $c = 5$, $\alpha:\beta:\gamma = 9:7:10$,

$$86939{\cdot}88\kappa^4 - 299457{\cdot}3056\kappa^2 + 46656 = 0,$$
$$\kappa_1 = 1{\cdot}8113113, \quad \kappa_2 \; 0{\cdot}4044374,$$
$$d_1 = 16{\cdot}301802, \quad e_1 = 12{\cdot}679179, \quad f_1 = 18{\cdot}113113,$$
$$d_2 = 3{\cdot}639937, \quad e_2 = 2{,}831062, \quad f_2 = 4{,}044374.$$

12164. (Professor Mandart.)—Soient un cercle O, un diamètre fixe AB et un point M variable sur le cercle. On mène un cercle C passant par le point O et tangent en M à la droite BM. Trouver les lieux géométriques décrits par les extrémités P et Q du diamètre du cercle C, qui est symétrique de AM par rapport à AB.

Solution by Profs. Droz-Farny, Mukhopadhyay, *and others.*



Le cercle C a évidemment son centre sur AM. Le diamètre PQ étant symétrique de AM par rapport à AB, la corde MO est parallèle à AB. Donc

$$\angle\, \text{MQO} = \text{QOA}$$
$$= \text{OMB} = \text{MBO}.$$

La figure MQOB est donc un parallélogramme; et, comme

$$\text{MQ} = \text{OB},$$

le lieu de Q est une circonférence égale à la circonférence O et de centre A.

Comme l'angle QMP est droit, le côté MP est perpendiculaire sur AB. De même PO est perpendiculaire sur QO; donc sur MB; par conséquent P est l'orthocentre du triangle isocele MOB. Or le côté OB de ce dernier étant fixe d'après un théorème bien connu, l'orthocentre P décrit une strophoïde ayant B comme sommet, O comme point double, et la tangente en A comme asymptote.

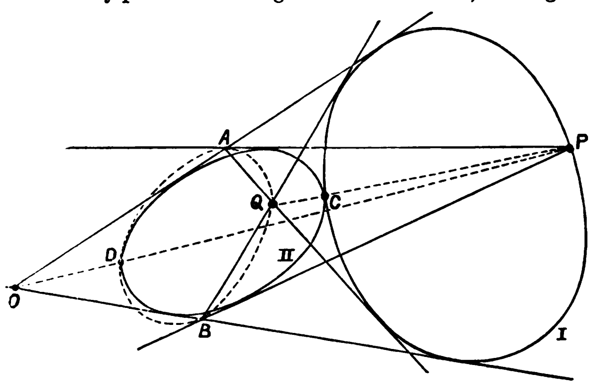
12203. (J. Griffiths, M.A.)—On the sides AB, BC of a triangle ABC describe circular segments, the first passing through the centre of

the circumcircle ABC, the second touching CA in C; prove that the point Q common to these segments lies on the Brocard circle of ABC. [Similar constructions with regard to the sides BC, CA; CA, AB will give points on the BROCARD-circle of ABC.]

Solution by R. TUCKER, M.A.

In "Note (iii.)" to the "Appendix" of Vol. XVII. of the *Lond. Math. Soc. Proc.* (January, 1887, p. 424), I have shown that the second circle, which, as is well known, passes through the positive Brocard point, intersects the Brocard circle on the "S" line through C. The coordinates of this point are $(ca, bc, 2ab \cos C)$. Now the equation to the first circle in the question is $2 \cos C \Sigma (\alpha\beta\gamma) = \gamma \Sigma (a\alpha)$, whence, &c. The points in the question are, as is well known, the angular points of Brocard's second triangle.

12205. (W. J. DOBBS, B.A. Suggested by Quest. 12162.)—OA, OB are common tangents to two conics, I and II, which touch one another at C. From any point P on I tangents are drawn to II, meeting OA and



OB in A and B respectively. From A and B are drawn two more tangents to I intersecting in Q. Prove that (1) P, C, Q are collinear; (2) the conic passing through Q, and touching PA and PB at A and B respectively, touches II at D; and (3) O, D, P are collinear.

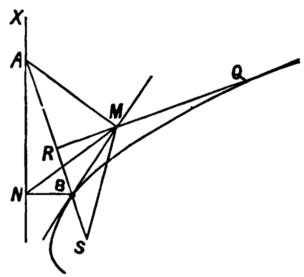
Solution by R. F. DAVIS, M.A.

Projecting the line AB to infinity, so that A, B become the imaginary circular points at infinity, we have the following simple theorem, which scarcely needs demonstration:—Two conics have a common focus O and

touch each other at C. Then, if Q, P be the other foci, and P lie on I, (1) C, Q, P are collinear; (2) the circle passing through Q, whose centre is at P, touches II, (3) at the extremity of the focal axis OP.

12198. (Professor LEINEKUGEL.)—Par un point quelconque M d'une tangente à une parabole P, en un point B on élève une perpendiculaire cette droite, qui rencontre la directrice en A; puis l'on trace AB. Démontrer que la perpendiculaire à AB, issue de M, est tangente à la parabole.

Solution by R. CHARTRES; H. W. CURJEL, B.A.; *and others.*



Draw BN perpendicular to the directrix. Let perpendicular from M on AB cut AB in R. Produce NA to X and RM to Q. Then

angles QMS, SMB
= BMQ = MBR, BRM
= ARM, RMA = MAX, BAN
= NBM, BMN
(from A, M, B, N are concyclic)
= MBS, BMS;

therefore QMS = MBS;

therefore RMQ touches the parabola.

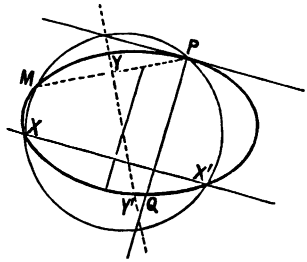
12212. (E. M. LANGLEY, M.A.)—If O, A, B, C, D, E be six concyclic points, prove that the projections of O on the pedal-lines of the five quadrilaterals BCDE, CDEA, DEAB, EABC, ABCD lie in a straight line.

Solution by Profs. STEGGALL, AIYAR, *and others.*

Draw OB_1, OC_1, OD_1, OE_1 perpendicular to AB, AC, AD, AE; then the pedal line of ABCD is easily seen to be that of $B_1C_1D_1$. For ABC, ABD, ACD are any three of the four triangles derivable from ABCD; their pedals are B_1C_1, C_1D_1, D_1B_1; and the feet of the perpendiculars from O on them are collinear. From O draw OC_2, OD_2, OE_2 perpendicular to B_1C_1, B_1D_1, B_1E_1; then C_2D_2 is the pedal of ABCD, D_2E_2 of ABDE, E_2C_2 of ABEC; but the feet of the perpendiculars from O on these lines are collinear; and ABCD, ABDE, ABEC are any three of the five quadrilaterals derivable from ABCDE. Hence all the quadrilaterals yield the same line, and we have a pedal of the pentagon. The reasoning can obviously be extended to polygons of any number of sides.

12114. (R. KNOWLES, B.A.)—PQ is a chord, normal at a fixed point P of a conic; XX′ are the ends of a chord parallel to the tangent at P. Prove that (1) the locus of the centre of the circle PXX′ is a straight line; (2) if this line meet the conic in YY′, P, Q, Y, Y′ are concyclic.

Solution by Prof. DROZ-FARNY; H. W. CURJEL, B.A.; *and others.*

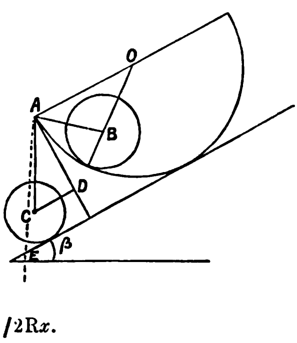


On sait que dans tout quadrilatère inscrit dans une conique les côtés opposés sont antiparallèles par rapport aux axes. Menons PM antiparallèle à XX′ par rapport aux axes; le quadrilatère PMXX′ sera inscriptible; or XX′ ayant une direction fixe, il en sera de même de PM et par conséquent le point M est fixe. Le lieu cherché est donc la perpendiculaire élevée sur PM en son point milieu.

Les droites YY′ et PQ respectivement perpendiculaires sur les antiparallèles MP et XX′ sont elles-mêmes antiparallèles par rapport aux axes, et le quadrilatère PQYY′ est par conséquent inscriptible.

5514. (A. MARTIN, LL.D.)—A hemispherical bowl, of radius R, is fixed to an inclined plane, of elevation β. Two spheres, of radii r, r_1, are connected by a string, of length a, and the first is put into the bowl and the other on the plane. Determine the position of the spheres when at rest.

Solution by H. J. WOODALL, A.R.C.S.



In the figure, let $AB = x$, $AO = R$; then $OB = R - r$. Draw AD tangential to semicircle at A, AE vertical, CD parallel to plane, and join BO.

$\cos OAB = \{R^2 + x^2 - (R-r)^2\}/2Rx$,

$\sin OAB = \cos BAD$

$= (1 - \cos^2 OAB)^{\frac{1}{2}}$.

Depth of B below A

$= x \cos BAE$

$= x \left[\cos\beta \left\{4R^2x^2 - [(R^2 + x^2) - (R-r)^2]^2\right\}^{\frac{1}{2}} - \sin\beta \left\{(R^2 + x^2) - (R-r)^2\right\}\right] / 2Rx.$

$AC = a - x, \quad \cos CAD = (R - r_1)/(a - x).$

Depth of C below A = AC cos CAE

$$= \cos\beta\,(\mathrm{R}-r_1)+\sin\beta\left\{(a-x)^2-(\mathrm{R}-r_1)^2\right\}^{\frac{1}{2}}.$$

Depth of centre of gravity of B and C below A,

$$= u = \Big(r^3\left[\cos\beta\left\{4\mathrm{R}^2x^2-[(\mathrm{R}^2+x^2)-(\mathrm{R}-r)^2]^2\right\}^{\frac{1}{2}}\right.$$
$$\left.-\sin\beta\left\{\mathrm{R}^2+x^2-(\mathrm{R}-r)^2\right\}\right]/2\mathrm{R}$$
$$+r_1^3\left[\cos\beta\,(\mathrm{R}-r_1)+\sin\beta\left\{(a-x)^2+(\mathrm{R}-r_1)^2\right\}^{\frac{1}{2}}\right]\Big)/(r^3+r_1^3).$$

Next find the maximum value of this expression by equating du/dx to zero and solving for x.

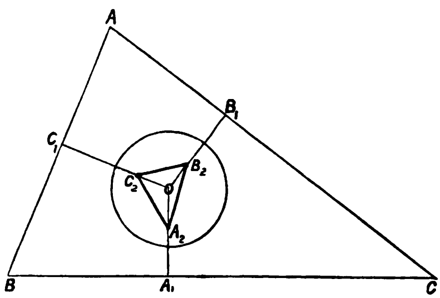
[For another solution, see Vol. xxix., p. 58.]

1173. (Editor.)—In a given triangle place (1) a given circle, so that the area of the triangle polar to the given triangle, relatively to the given circle, may be a minimum; also, find (2) the locus of the centre of the circle when the area of the polar triangle is constant.

Solution by Profs. Lampe, Chakrivarti, *and others.*

Let $OA_1 = x$, $OB_1 = y$, $OC_1 = z$ be perpendicular respectively to $BC = a$, $CA = b$, $AB = c$, r = radius of the given circle. Then, taking

$OA_1 \,.\, OA_2 = r^2$,
$OB_1 \,.\, OB_2 = r^2$,
$OC_1 \,.\, OC_2 = r^2$,



$A_2B_2C_2$ is the triangle polar to the given triangle relatively to the given circle; whence (putting R = circum-radius) we get at once its area to be

$$A_2B_2C_2 = \frac{r^4}{4\mathrm{R}xyz}(ax+by+cz) = \frac{r^4\Delta}{2\mathrm{R}xyz},$$

Δ being the area of ABC.

To find the minimum of $A_2B_2C_2$, we must find the maximum of

$$f(x, y, z) = xyz - \lambda\,(ax+by+cz-2\Delta).$$

Equating the partial differential quotients to 0, we obtain

$$yz-\lambda a = 0,\quad zx-\lambda b = 0,\quad xy-\lambda c = 0,$$

or

$$xa = yb = zc = \lambda^{\frac{1}{2}}(abc)^{\frac{1}{2}},$$

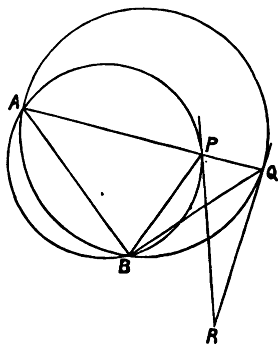
that is to say, O is the centroid of the triangle. Moreover

$$\lambda^{\frac{1}{2}} = \tfrac{2}{3}\Delta/(abc)^{\frac{1}{2}},\ A_2B_2C_2 = 27r^4/4\Delta,\ \text{is the minimum of } A_2B_2C_2.$$

2. If $A_2B_2C_2$ has the constant area q^2, then $xyz = \dfrac{r^4\Delta}{2Rq^2}$ is the equation of the locus of O.

12195. (Professor Droz-Farny.) — Deux coniques se coupent en A, B, C, D. Une transversale quelconque par A coupe les coniques en P et Q. Les tangentes en P et Q, la transversale APQ, et les côtés du triangle BCD, sont tangentes à une même conique.

Solution by W. J. Dobbs, B.A.; H. W. Curjel, B.A.; *and others.*

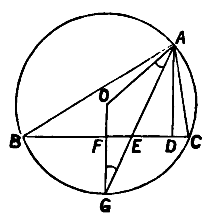


Project CD into the circules; then the two conics become circles through A, B. Let the tangents at P and Q cut in R. Then $\angle$ PRQ = the difference between the angles in the segments ABQ, ABP = $\angle$ PBQ;

therefore B, P, Q, R are concyclic; therefore APQ, PR, QR touch a parabola with focus B, *i.e.*, touch a conic touching the sides of ΔBCD.

11986. (Prof. Rangasawmi Iyengur.)—Given the altitude, the bisector of the vertical angle, and the median of a triangle, construct it.

Solution by R. Chartres; W. J. Dobbs, B.A.; *and others.*



Let AD, AE, AF, be the altitude, the bisector of the vertical angle, and the median respectively.

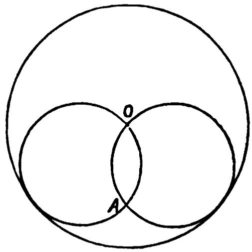
AE meets the perpendicular from F at G, a point on the circumcircle. Make

$$\angle \text{GAO} = \text{AGO},$$

then O is the centre of the circumcircle, and BAC the triangle required. The proof is obvious.

1268. (S. Watson.)—Two points A, B are taken at random upon the surface of a given circle, of centre O; find the chance that the circle through O, A, B shall be less than one-fourth of the given circle.

Solution by H. W. Curjel, B.A.



Draw two circles on radii of the given circle as diameters and passing through A. Then if B is inside one of these two circles and outside the other, the circle OAB will be less than $\frac{1}{4}$ the given circle, and, if not, it will be greater; therefore the required chance

$$= \frac{\int_0^a 2\left\{\frac{\pi a^2}{4} - \frac{1}{2}\left[a^2 \sin^{-1} r/a - r(a^2-r^2)\ \right]\right\} \times 2\pi r\, dr}{\pi a^2 \cdot \pi a^2}$$

$$= \frac{\frac{\pi a^4}{2} - \left[a^2 r^2 \sin^{-1} r/a\right]_0^a + \int_0^a \frac{a^2 r^2}{(a^2-r^2)^{\frac{1}{2}}}\, dr + \int_0^a \frac{2a^2r^2 - 2r^4}{(a^2-r^2)^{\frac{1}{2}}}\, dr}{\pi a^4}$$

$$= \frac{\frac{\pi a^4}{2} - \frac{\pi a^4}{2} + \int_0^a \frac{3a^2r^2 - 2r^4}{(a^2-r^2)^{\frac{1}{2}}}\, dr}{\pi a^4} = \frac{1}{\pi a^4}\, a^4 \int_0^{\frac{1}{2}\pi} (3\sin^2\theta - 2\sin^4\theta)\, d\theta = \frac{3}{8},$$

where $\sin\theta = r/a$.

11920. (R. Knowles, B.A.)—A circle having a fixed centre cuts a given conic in ABCD. G is the point of intersection of the diagonals, and EF the third diagonal of the quadrilateral. Prove that (1) the locus of the points E, F, G is a rectangular hyperbola; (2) the envelope of EF, EG, and FG is a parabola.

Solution by Prof. Droz-Farny; W. J. Dobbs, B.A.; *and others.*

Représentons par P et O respectivement les centres du cercle et de la conique. Le lieu des centres des coniques du faisceau ABCD est une hyperbole qui passe par les points milieux des 6 cordes communes, par P et O et par les sommets du triangle conjugué EFG. Ses directions asymptotiques, comme un des éléments du faisceau est un cercle, sont parallèles aux axes de la conique O′. L'hyperbole est donc équilatère.

La conique d'Apollonius du point P par rapport à la conique O a les mêmes directions asymptotiques; elle passe par P et O et d'après Chasles par les points milieux des 6 cordes communes. Les 2 hyperboles se confondent donc, et par conséquent, lorsque les cercles varient, le lieu des points EFG est l'hyperbole d'Apollonius du point P par rapport à la conique O.

Le triangle EFG etant conjugué au cercle, les quatre points PEFG

forment un quadruple orthocentrique, et par conséquent la circonférence EFG passe par le point P′ diamétralement opposé à P sur l'hyperbole.

Le triangle EFG étant aussi conjugué à la conique, sa circonférence circonscrite coupe, d'après un théorème de Fayre, orthogonalement le cercle décrit sur le grand axe de O comme diamètre; le cercle EFG passera donc, par un deuxième point fixe π, le conjugué harmonique de P par rapport aux extrémités du diamétre PO du cercle principal considéré.

Les côtés du triangle EFG sont par conséquent les cercles d'intersection d'une hyperbole fixe avec les cercles d'un faisceau ayant un de ses centres sur l'hyperbole. Ils enveloppent donc une conique qui doit être une parabole, car le cercle singulier du faisceau est constitué par la droite $P\pi$ et la droite infinie. Or cette dernière, étant une cercle commune à ce cercle et à l'hyperbole, est une tangente à la conique, qui est donc bien une parabole.

12127. (Professor Nilkantha Sarkar.)—A rod is marked in four points at random. A. bets B. £50 even that no segment exceeds $\frac{7}{16}$ of the whole. Prove that A.'s expectation is 3*s*. 10*d*. nearly.

Solution by Professors Zerr, Mukhopadhyay, *and others.*

$$\begin{aligned}\text{Chance} &= \left(\frac{35}{16}-1\right)^4-5\left(\frac{28}{16}-1\right)^4+10\left(\frac{21}{16}-1\right)^4\\ &= \left(\frac{19}{16}\right)^4-5\left(\frac{12}{16}\right)^4+10\left(\frac{5}{16}\right)^4=\frac{32891}{65536}\\ &= 1-5\left(1-\frac{7}{16}\right)^4+10\left(1-\frac{14}{16}\right)^4=1-5\left(\frac{9}{16}\right)^4+10\left(\frac{2}{16}\right)^4\\ &= \frac{32891}{65536}\text{ of £}100=\text{£}50{\cdot}187683=3s.\ 904392d.\end{aligned}$$

12189. (Professor Morley.)—A circle A rolls on a circle B of half the curvature; any point of A describing an ellipse. Regard this ellipse as rigidly attached to B. Prove that when B rolls on A, which is fixed, the ellipse passes through a fixed point. Generalize.

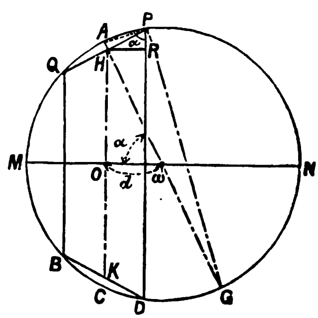
Solution by H. W. Curjel, B.A.; H. J. Woodall, A.R.C.S.; *and others.*

Let a point P, carried by a curve A, which rolls on a curve B, describe a curve Q. When B rolls on A the relative motion of the two figures is the same as when A rolls on B. Hence, when A rolls on B, Q being fixed to B, the curve Q will always pass through P, if the relative position of the curves A and B is at any time the same as their relative position at any time when A was rolling on B. Similarly, with a similar

condition, it may be shown that if a curve p carried by A envelopes the curve Q when A rolls on B, the curve Q will always touch p when B rolls on A, Q being fixed to B.

8466. (By A. GORDON.)—Given a sphere, radius r; find the locus of centre (*i.e.*, the mid-point of its intercepted axis) and the corresponding dimensions of the cylinder which *pierces* and cuts off a given area λr^2 and is of given volume μr^3.

Solution by Profs. BORDAGE, HANUMANTA, *and others.*



The surface determined by the intersection of the cylinder with the sphere (that is to say, the surface λr^2 cut off by the cylinder) is formed by two equal spherical zones QAB and BCD. This surface is equal to

$$2 \times 2\pi r \,.\, \mathrm{AH} = 4\pi r \,.\, \mathrm{AH} = \lambda r^2.$$

Then, $\mathrm{AH} = \dfrac{\lambda r^2}{4\pi r} = \dfrac{\lambda r}{4\pi}$. Now

$$\mathrm{HO} = \mathrm{H}\omega \sin\alpha = (r - \mathrm{AH})\sin\alpha$$

$$= \left(r - \frac{\lambda r}{4\pi}\right)\sin\alpha = \frac{r(4\pi-\lambda)\sin\alpha}{4\pi}.$$

Then, the altitude HC of the cylinder QPBD is $\dfrac{r(4\pi-\lambda)\sin\alpha}{2\pi}$.

In the triangle APR (the angle PRH = 90°), we have $\mathrm{HR} = \mathrm{HP}\sin\alpha$, and in the triangle APG (the angle APG is right, because AG perpendicular on QP is a diameter), we have

$$(\mathrm{PH})^2 = \mathrm{AH} \times \mathrm{HG} = \mathrm{AH}(2r - \mathrm{AH}) = \frac{\lambda r^2(8\pi-\lambda)}{16\pi^2},$$

$$\mathrm{PH} = \frac{r\sqrt{\lambda(8\pi-\lambda)}}{4\pi}, \qquad \mathrm{HR} = \frac{r\sin\alpha\sqrt{\lambda(8\pi-\lambda)}}{4\pi}.$$

But HR is the radius of the cylinder; the volume of this cylinder is

$$\pi\,.\,(\mathrm{HR})^2\,.\,\mathrm{HC} = \frac{\lambda\,.\,r^3\sin^3\alpha(8\pi-\lambda)(4\pi-\lambda)}{32\pi^2}; \quad \text{hence}$$

$$\frac{\lambda\,.\,r^3\sin^3\alpha(8\pi-\lambda)(4\pi-\lambda)}{32\pi^2} = \mu r^3, \text{ or } \lambda\sin^3\alpha(8\pi-\lambda)(4\pi-\lambda) = 32\pi^2\mu.$$

Now, it is easy to obtain the dimensions of the cylinder, for

$$\mathrm{HC} = \frac{r(4\pi-\lambda)\sin\alpha}{2\pi}; \quad \mathrm{HR} = \frac{r\sin\alpha\sqrt[2]{\lambda(8\pi-\lambda)}}{4\pi};$$

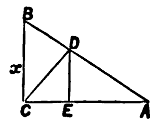
and the distance $O\omega = d$ of the centre O of the cylinder to the centre ω of the sphere, which determines the position of the point O, is given by

$$d = H\omega \,.\, \cos\alpha = \frac{r(4\pi - \lambda)}{4\pi}\,.\cos\alpha.$$

Sin α being known, cos α is also known ; then it is easy to calculate the distance d. If $\sin^2\alpha$ is replaced by its value, d is obtained.

1239. (EDITOR.)—Given one side of a right-angled triangle ; construct it, so that the difference between the other side and the adjacent segment of the hypotenuse, cut off by a perpendicular from the right angle, may be a *maximum*. Prove that the perpendicular divides the hypotenuse in extreme and mean ratio, and that the greatest segment is equal to the remote side of the triangle.

Solution by R. CHARTRES ; Prof. IYENGUR ; *and others.*



Let ACB be a right-angled triangle, CD a perpendicular on AB, then

$$BC - BD, \text{ or } x - \frac{x^2}{(1+x^2)^{\frac{1}{2}}} \text{ is to be a maximum,}$$

or $$x^2(1+x^2) = 1 ; \quad \therefore\; x = \frac{1}{(1+x^2)^{\frac{1}{2}}}.$$

Hence BC = AD, and rectangle AB . BD = AD^2.

Construction.—Divide AC at E, so that AC . CE = AE^2 ; draw a semicircle on AC, meeting the perpendicular ED at D : join AD, and produce to meet perpendicular CB at B.

3646. (HUGH MCCOLL, B.A.)—Find (by STURM's Theorem or otherwise) the number of real roots of the equation

$$x^6 - 19x^5 + 90x^4 + 11x^3 - 99x^2 + 170x - 70 = 0 ;$$

and determine the limits between which each real root is situated.

Solution by H. J. WOODALL, A.R.C.S.

We find, by the usual rule for obtaining STURM's Functions,

$$x^6 - 19x^5 + 90x^4 + 11x^3 - 99x^2 + 170x - 70 \equiv (x - 3\cdot16667)$$
$$\times (x^5 - 15\cdot83333x^4 + 60x^3 + 5\cdot5x^2 - 33x + 28\cdot33333) - 3\cdot16667 \times 6\cdot35965$$
$$\times (x^4 - 9\cdot70758x^3 + 2\cdot41241x^2 - 1\cdot84551x - 0\cdot97931) ;$$

so, $f_1(x) \equiv (x-6\cdot12575)f(x) - 6\cdot12575$
$$\times\, 0\cdot30668\,(x^3-11\cdot7766x^2+23\cdot0624x-11\cdot8885),$$
$f_2(x) \equiv (x+2\cdot0690)f_3(x) - 2\cdot0690 \times 1\cdot7962\,(-x^2+10\cdot1372x-5\cdot7985),$
$f_3(x) \equiv (-x+1\cdot6393)f_4(x) - 1\cdot6393 \times 0\cdot3935\,(-x+3\cdot69),$
$f_4(x) \equiv (+x-6\cdot43)f_5(x) + 6\cdot43 \times 2\cdot7.$

Then we obtain the following table:—

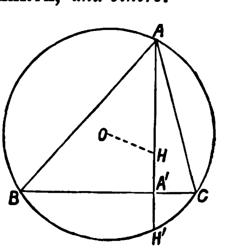
$x =$	$+\infty$	$+10$	$+9$	$+1$	0	-1	-2	$-\infty$
$f(x)$	+	+	+	+	−	−	+	+
$f_1(x)$	+	+	−	+	+	−	−	−
$f_2(x)$	+	+	−	−	−	+	+	+
$f_3(x)$	+	+	−	+	−	−	−	−
$f_4(x)$	−	−	+	+	−	−	−	−
$f_5(x)$	−	−	−	+	+	+	+	+
$f_6(x)$	−	−	−	−	−	−	−	−
Changes of sign	1	1	3	3	4	4	5	5

Therefore there are four real roots, which lie between +1 and 0, between −1 and −2, and two between 9 and 10; they are 0·527528 and −1·477225, 9·4721 and 9·4772. The other two roots are imaginary.

[*Otherwise*:—Applying the method of his article in Vol. IV., the PROPOSER finds one real root between 0 and 1, two real and nearly equal roots between 9 and 10 (9·4721 ... and 9·4772 ...), one between −1 and −2, and two imaginary roots.]

12061. (Professor VERRINA.)—Parmi tous les triangles acutangles inscrits dans un même cercle, quel est celui pour lequel le produit des segments determinés sur une hauteur par le point de rencontre des deux autres est maxima?

Solution by Profs. DROZ-FARNY, BHATTACHARYA, *and others.*



Soit AA′ une hauteur du triangle ABC; H l'orthocentre, et H′ le deuxième point de coupe de la hauteur avec la circonférence circonscrite.

La puissance du point H par rapport à la circonférence est évidemment égale à

$$R^2-(OH)^2.$$

On a donc:

$$AH\,.\,HH' = R^2-(OH)^2;$$
$$HH' = 2HA',$$
$$AH\,.\,HA' = \tfrac{1}{2}(R^2-OH).$$

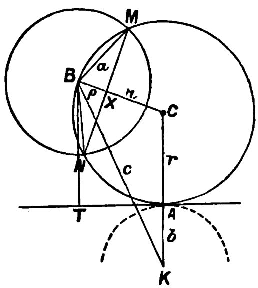
Le maximum du produit aura lieu pour (OH) = 0. L'orthocentre coincidant avec le centre du cercle circonscrit, le triangle est équilateral.

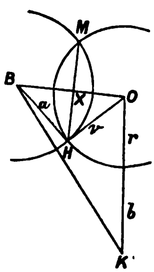
12156. (Professor CLAYTON, B.A.)—(1) A variable circle touches a given line A, and passes through the centre B of a given circle. If MN be its chord of intersection with the circle, and BX be drawn perpendicular to MN, prove that (1) the locus of X is a cardioid: (2) if the line A be replaced by a circle, all other circumstances being unaltered, the locus of X is of the form $\zeta = A + B\cos\theta$; and (3) if the variable circle touch the circle A, and cut the circle B orthogonally, the locus of X is a bicircular quartic.

Solution by C. MORGAN, M.A., R.N.; Prof. DROZ-FARNY; *and others.*

1. Let $BX = \rho$, $XBT = \theta$, $OA = r$, $BM = a$.

$BT = r(1+\cos\theta)$ and $\rho . 2r = a^2$; $\therefore\ \rho = \dfrac{a^2}{2BT}(1+\cos\theta)$.





2. Let K be the centre of the circle A, and b its radius.

Let $BX = \rho$, $XBK = \theta$, $BK = c$;

then $\cos\theta = \dfrac{r^2+c^2-(r+b)^2}{2rc}$; whence $\rho = \dfrac{a^2}{c^2-b^2}(c\cos\theta+b)$.

3. $r^2+a^2 = OB^2$ and $\rho . OB = a^2$; $\therefore\ \cos\theta = \left(\dfrac{a^2+c^2-2rb-b^2}{2a^2b}\right)\rho$;

$$\therefore\ 2a^2c\cos\theta = \rho(a^2+c^2-b^2) - 2ab(a^2-\rho^2)^{\frac12}.$$

1340. (W. S. B. WOOLHOUSE, F.R.A.S., F.S.S.)—Determine the value of the expression

$$(-1)^{\frac12}\sin\left[(-1)^{\frac12}\log_e\left\{x+(x^2+1)^{\frac12}\right\}\right]\cos\left[(-1)^{\frac12}\log_e\left\{x+(x^2-1)^{\frac12}\right\}\right].$$

Solution by R. CHARTRES.

Let $\qquad x+(x^2-1)^{\frac{1}{2}} = y, \quad \text{or} \quad x = \frac{1}{2}(1/y+y);$

therefore $\quad \cos\left[i\log\left\{x+(x^2-1)^{\frac{1}{2}}\right\}\right] = \frac{1}{2}\left(e^{-\log y}+e^{\log y}\right) = x,$

similarly, $\quad i\sin\left[i\log\left\{x+(x^2+1)^{\frac{1}{2}}\right\}\right] = -x;$

therefore product $= -x^2$.

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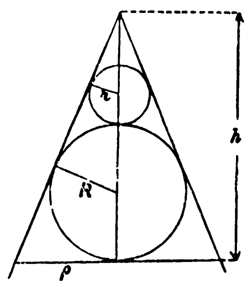
756. (MATTHEW COLLINS, B.A.)—If R, r be the radii of two spheres inscribed in a cone, so that the greater (R) may touch the less and also the base of the cone, find the volume of the cone.

Solution by H. W. CURJEL, B.A.



Let h be the height of the cone, and ρ = the radius of the base; then we have

$$\frac{h-R}{R} = \frac{h-(2R+r)}{r}; \quad \therefore\ h = \frac{2R^2}{R-r};$$

$$\frac{\rho^2}{h^2} = \frac{R^2}{(h-R)^2-R^2} = \frac{R^2}{h^2-2Rh};$$

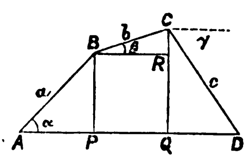
$$\therefore\ \rho^2 = \frac{hR^2}{h-2R} = \frac{2R^4}{2R^2-2R^2+2Rr} = \frac{R^3}{r};$$

therefore volume of cone

$$= \tfrac{1}{3}\pi\rho^2 h^2 = \tfrac{2}{3}\pi\,\frac{R^5}{(R-r)\,r}.$$

4030. (W. BARLOW.)—Construct a quadrilateral when three sides are given and the fourth is trisected by perpendiculars from the opposite angles.

Solution by H. J. WOODALL, A.R.C.S.



Since $\qquad CQ = CR+BP;$

therefore $\quad c\sin\gamma = a\sin\alpha + b\sin\beta.$

Since $\qquad AP = PQ = QD;$

therefore $\quad a\cos\alpha = b\cos\beta = c\cos\gamma;$

from these, eliminating β and γ, we get

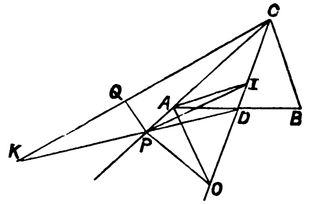
$$3a^2\cos^2\alpha$$

$$= a^2+b^2+c^2 - 2\left\{a^4+b^4+c^4-(b^2c^2+c^2a^2+a^2b^2)\right\}^{\frac{1}{2}},$$

the plus sign being inadmissible, whence we can solve the problem.

482. (J. H. SWALE.)—In the triangle ABC, take I and O, the centres of the inscribed and escribed circle touching the sides CA and CB produced; draw OP perpendicular to CA, also CQ parallel to PI, meeting OP at Q; then prove that (1) 2PQ = the perpendicular CE; and (2) if CO cut AB in D, and DP be drawn to meet CQ in K, then QK = QC.

Solution by W. J. GREENSTREET, M.A.



1. $QP/OP = CI/OI = CI \sin \frac{1}{2}B/AI$
$= \sin \frac{1}{2}B \sin \frac{1}{2}A/\sin \frac{1}{2}C;$

$\therefore$ $QP = s \sin \frac{1}{2}B \sin \frac{1}{2}A/\sin^2 \frac{1}{2}C$
$= \Delta/c = \frac{1}{2}CE,$

or $2QP = CE.$

2. Again, $PI/QC = OI/OC,$
$PI/CK = DI/DC;$

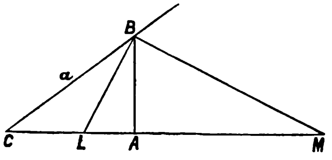
$\therefore$ $CK/CQ = OI.DC/(OC.DI),$

$$\frac{DC}{OC} = \frac{2(s-c)}{a+b}; \quad \frac{ID}{OD} = \frac{r}{r_c}; \quad \therefore \quad \frac{OI}{ID} = \frac{r+r_c}{r} = \frac{a+b}{s-c};$$

therefore $\dfrac{OI.DC}{OC.DI} = 2,$ *i.e.*, $CK/CQ = 2$, or $CQ = QK$.

12023. (Professor BIGOT.)—Construire un triangle, connaissant un côté, un angle adjacent, et le rapport de la surface de ce triangle à celle du triangle formé par les deux bissectrices de l'angle donné et le côté opposé. Examiner en particulier le cas où le rapport est égal à 2.

Solution by J. M. STOOPS, B.A.; Prof. CHAKRIVARTI; *and others.*



Let B be the given angle, a the given side, and BL, BM the bisectors of B; then

$$\frac{AL}{b} = \frac{c}{a+c},$$
$$\frac{AM}{b} = \frac{c}{a-c};$$

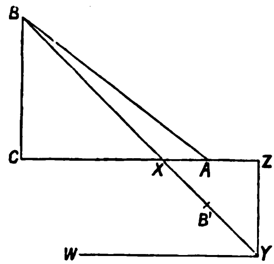
therefore $\dfrac{LM}{b} = \dfrac{2ac}{a^2-c^2}$, $\dfrac{\Delta ABC}{\Delta LBM} = K = \dfrac{a^2-c^2}{2ac};$

therefore $c = a\left\{-K + (1+K^2)^{\frac{1}{2}}\right\};$

thus c is given, which fixes the point A. When $K = 2$, $c/a = \sqrt{5}-2$.

12181. (I. Arnold.)—Given the difference between the hypotenuse and each leg of a right-angled triangle, to construct the triangle.

Solution by Gertrude Poole, B.A.; H. W. Curjel, B.A.; *and others.*

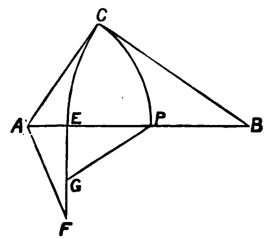


Let ZA, ZX be the given differences; let ZA be < ZX; and let Z, A, X be in a straight line. Draw ZY at right angles to and equal to ZX; through Y draw YW parallel to ZX; find B, B′, the centres of the circles touching YZ, YW, and passing through A, B being the more remote from Y; and from B draw BC perpendicular to XZ. Then ABC is evidently the triangle required;

since $\quad AB = CZ,$

and $\quad BC = CX.$

[Mr. Biddle gives the following Solution:—



Let AE, EF be the respective differences, placed at right angles to each other in E. Produce the shorter indefinitely, and join AF. In EF, take FG = AE, and make GP = AF, and PB = EF. Then, with centre A, radius AP, and with centre B, radius BE, describe circles intersecting in C, and join AC, BC. ABC is the required triangle.

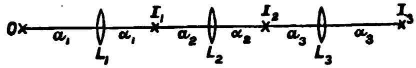
For, suppose ABC to be any right-angled triangl , and AP = AC, BE = BC. Then $(EP+AE)^2+(EP+BP)^2$
$$= (EP+AE+BP)^2,$$
whence $EP^2 = 2AE \cdot BP$ = twice the product of the differences. But, in the present instance,

$$EP^2 = GP^2 - GE^2 = EF^2 + AE^2 - (EF-AE)^2 = 2AE \cdot EF$$
= twice the product of the given differences,

of which one is AE, and the other EF = PB by construction.]

5096. (The Rev. T. R. Terry, M.A.)—A convex lens is held so that the distance between a bright point and its image is the least possible; two other lenses are then introduced, one half-way between the first lens and the luminous point, the other half-way between the first lens and the image of the point. If the position of the image remains unaltered, the sum of the focal lengths of the three lenses will be zero.

Solution by Professor Lampe.



Let L_1, L_2, L_3 be three lenses with the focal lengths p_1, p_2, p_3 respectively: O an object (a bright point); I_1 the image of O formed by L_1; I_2 the image of I_1 produced by L_2; finally I_3 the image of I_2 generated by L_3; d_1 = distance of L_1 and L_2; d_2 = distance between L_2 and L_3. Then (*vide* figure) the equations will hold:

$$\left.\begin{array}{c} 1/a_1 + 1/\alpha_1 = 1/p_1, \quad 1/a_2 + 1/\alpha_2 = 1/p_2, \quad 1/a_3 + 1/\alpha_3 = 1/p_3 \\ \alpha_1 + a_2 = d_1, \quad \alpha_2 + a_3 = d_2 \end{array}\right\} \quad \ldots\ldots(1).$$

In Question 5096, L_2 is supposed to be in such a position as to make $OL_2 + L_2I_3$ a minimum if I_3 is the image of O produced by L_2 alone, L_1 and L_3 being removed. Therefore $OL_2 = L_2I_3 = 2p_2$, and consequently $OL_1 = L_1L_2 = L_2L_3 = L_3I_3 = p_2$. Whence the equations (1), writing $a_1 = \alpha_3 = p_2$, become

$$\left.\begin{array}{c} 1/p_2 + 1/\alpha_1 = 1/p_1, \quad 1/a_2 + 1/\alpha_2 = 1/p_2, \quad 1/a_3 + 1/p_2 = 1/p_3 \\ \alpha_1 + a_2 = p_2, \quad \alpha_2 + a_3 = p_2 \end{array}\right\} \quad \ldots\ldots(2).$$

Eliminating the four quantities $\alpha_1, a_2, \alpha_2, a_3$ from the five equations (2), we get successively

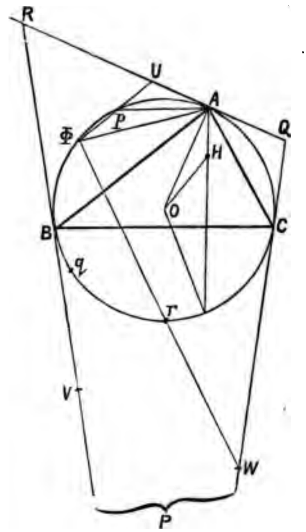
$$\alpha_1 = \frac{p_1 p_2}{p_2 - p_1}, \quad a_3 = \frac{p_2 p_3}{p_2 - p_3}, \quad a_2 = \frac{p_2^2 - 2p_1 p_2}{p_2 - p_1}, \quad \alpha_2 = \frac{p_2^2 - 2p_2 p_3}{p_2 - p_3},$$

$$\frac{p_2 - p_1}{p_2^2 - 2p_1 p_2} + \frac{p_2 - p_3}{p_2^2 - 2p_2 p_3} = \frac{1}{p_2}, \quad \text{or} \quad p_2 = p_1 + p_3.$$

12222. (Professor Genese, M.A.)—Each of the four triangles formed by the common tangents to two conics is homologous with each of the four triangles whose vertices are the points of intersection of the conics; or, if more than one conic can be simultaneously described about one triangle and inscribed in another triangle, the two triangles are homologous.

Solution by Professors Droz-Farny, Emmerich, *and others.*

Représentons par A, B, C, D les quatre points d'intersection des deux coniques, et par α, β, γ, δ leurs quatre tangentes communes; de manière à ce que, AB passant par le sommet x du triangle conjugué aux deux coniques, les tangentes α et β se croisent sur le côté opposé yz. On sait qu'il existe une troisième conique telle que les sommets A, B, C, D soient respectivement les poles des tangentes α, β, γ, δ. Or deux triangles polaires réciproques par rapport à une conique sont homologiques.



12250. (R. F. DAVIS, M.A.)—If ABC be a triangle whose circumcentre is O and orthocentre H, and if upon Ap, the chord of the circumcircle parallel to BC, a triangle $A\Phi p$ be described directly similar to the triangle OAH, prove that Φ is the point of contact of the nine-point and inscribed circles of the triangle PQR formed by drawing tangents at A, B, C to the circumcircle of ABC. The order of the proof is as follows:—(1) Φ lies on the circumcircle of ABC. (2) The triangles $B\Phi q$, $C\Phi r$ are similar to OBH, OCH respectively, where Bq, Cr are chords parallel to CA, AB. (3) The sides of the triangle pqr are parallel to those of PQR. (4) The triangles pqr, UVW are similar and similarly situated, the ratio of similitude being $= 1 - OH^2/OA^2$ $(= 8 \cos A \cos B \cos C)$. (5) The point Φ lies on the circumcircle of UVW. (6) The tangents at Φ to the two circles are coincident. The point Φ has its distances from A, B, C proportional to Ap, Bq, Cr [*i.e.* to $\sin(B \backsim C)$, $\sin(C \backsim A)$, $\sin(A \backsim B)$], respectively, and consequently its trilinear coordinates inversely proportional to these quantities.

Solution by H. W. CURJEL, B.A.; Professor LAMPE; *and others.*

Let AH cut the circumcircle of ABC in K; then

$$\angle p\Phi A = \angle OAH = C - B;$$

therefore Φ is on the circle ABC; therefore $\angle r\Phi C = \angle OCH$.

Now $\angle CHK = B$, and

$$\angle OHK = \angle \Phi CA = \angle \Phi CH + (\tfrac{1}{2}\pi - A) = \angle O\Phi r + (\tfrac{1}{2}\pi - A),$$

for $$\angle O\Phi C = OC\Phi;$$

$$\therefore\ \angle OHC = \angle Or\Phi + B + \tfrac{1}{2}\pi - A = \angle Or\Phi + ORC = \angle \Phi rC.$$

Hence the triangles $C\Phi r$, OCH are similar, and in the same manner triangles $B\Phi q$, OBH are similar.

Again, arc Bp = arc AC = arc Br; therefore pr is parallel to PR; and, similarly, the other sides of Δpqr are parallel to those of ΔPQR.

Again, triangles OHK, UpA are similar; hence

$$UA : Up = OK : OH, \quad U\Phi : Up = OA^2 : OH^2.$$

Hence, by symmetry, the sides of ΔUVW are parallel to the sides of Δpqr, and therefore to those of ΔPQR, and the O' in ΦO produced (such that $O'\Phi : OO' = OK^2 : OH^2$) is the centre of the circle UVW, which

evidently also passes through Φ, and therefore touches circle ABC in Φ. But U, V, W are evidently the middle points of the sides of the $\triangle$PQR, and thence this circle is the nine-point circle of $\triangle$PQR.

The ratio of similitude of the triangles pqr, UVW is evidently

$$\Phi p/\Phi U = 1 - OH^2/OA^2.$$

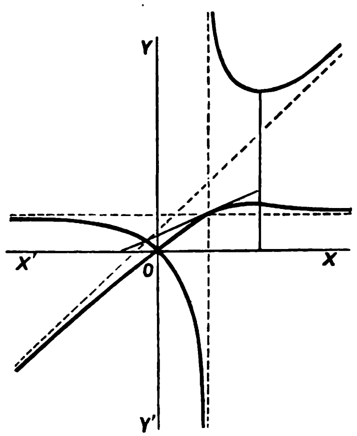
The trilinear coordinates of Φ are inversely proportional to sin (B − C), sin (C − A), sin (A − B).

[The PROPOSER remarks that the Question really implies another proof of FEUERBACH's theorem that the inscribed and nine-point circles of any triangle touch each other.]

12201. (EDITOR.)—Trace the locus of the equation

$$(x-a)\,y^2 = (y-b)\,x^2.$$

Solution by W. J. DOBBS, B.A.; Professor BHATTACHARYA; *and others.*



It is easily seen that the cubic has three real asymptotes, namely, $x = a$, $y = b$, and $y = x + a - b$. It meets the first two in the point (a, b). Transferring the axes to this point, we see that the equation to the tangent at the origin is $y/x = b^2/a^2$. This line must intersect the remaining asymptote in a point on the cubic.

For all real points on the curve, $x-a$ and $y-b$ must be of the same sign. This shows that no part of the curve lies in two of the quadrants formed by the first two mentioned asymptotes.

We shall suppose that a and b are positive quantities, and $a > b$. The origin is a multiple point, the equation to the tangents at the origin being $ay^2 = bx^2$. The curve does not intersect the axes except at the origin and at infinity. Arranging the equation as a quadratic in y, we see that y has two real values for every value of x; that these values are both positive as long as $x-a$ is positive, and of opposite signs when $x-a$ is negative.

Arranging the equation as a quadratic in x, we see that y is real only so long as $(y-2a)^2$ is $> 4a\,(a-b)$. When y is outside these limits, it has two values which are both positive so long as $y-b$ is positive, and of opposite signs when $y-b$ is negative.

We have $$\frac{dy}{dx} = \frac{2(y-b)x-y^2}{2(x-a)y-x^2} = \frac{(x-2a)y^3}{(y-2b)x^3}.$$

Hence dy/dx vanishes when $x = 2a$, and is not infinite for any real point on the curve at a finite distance.

Thus the curve is as shown in the accompanying diagram, the dotted lines being the asymptotes.

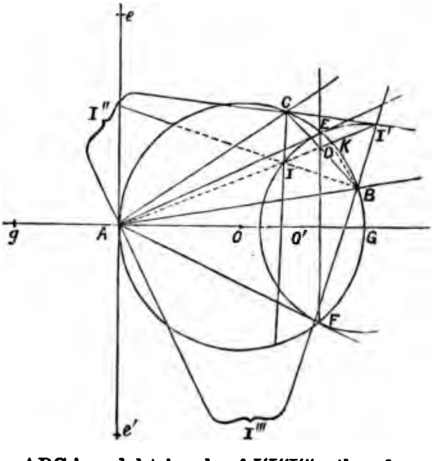
12197. (Professor AUBERT.)—Par un point fixe A d'une circonférence donnée, on mène deux cordes AB et AC, dont le produit a une valeur constante m^2; puis on joint BC. Démontrer que (1) le lieu du pied D de la bissectrice de l'angle BAC du triangle ABC est une droite; (2) le centre I du circle inscrit et le centre I′ du cercle exinscrit dans l'angle BAC décrivent une même circonférence; (3) les centres I″, I‴ des deux autres cercles exinscrits décrivent aussi une même circonférence.

Solution by R. TUCKER, M.A.; H. W. CURJEL, B.A.; *and others.*

From properties of the triangle, we have

$$m^2 = AB \, . \, AC = BD \, . \, DC + AD^2 = AD \, . \, DK + AD^2 = AD \, . \, AK;$$

hence locus of D is inverse of locus of K, *i.e.*, in this case, since A is on the circle, a straight line EDF perpendicular to AO ....(1).



Again, since ABC is pedal triangle of I′I″I‴; therefore

$$AI \, . \, AI' = AB \, . \, AC = m^2 = AE^2,$$

and I, I′ lie on a circumference, centre G, radius GE ....(2).

Lastly, $\quad AI''.AI''' = AB.AC = AE^2.$

The images of I″, I‴, with respect to AO, which correspond to the image of ABC, and I″, I‴, evidently lie on a circle which cuts the tangent at A in e, e', such that $Ae = Ae' = AE$. Hence the locus of I″, I‴ is a circle cutting OA produced in g, so that $Ag = GE$. If O′ is the centre of this circle, then $O'g = 2OA^2/GE$.

12220. (Professor LAMPE, LL.D.)—Let $P(x_1, y_1)$ be a point of the ellipse $x^2/a^2 + y^2/b^2 = 1$, C the centre of the circle osculating at P. There are two normals CF_1, CF_2, distinct from CP, which may be drawn from C to the points F_1, F_2 of the ellipse. (1) The line joining F_1F_2 has the equation $x/x_1 + y/y_1 + 1 = 0$, to be used for a construction of C. (2) F_1F_2 envelopes the curve $(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1$.

Solution by W. J. DOBBS, B.A.; R. H. W. WHAPHAM, M.A.; *and others.*

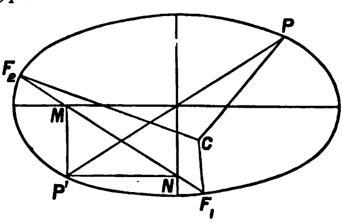
1. The normals at the intersections of the ellipse with $lx + my = 1$ and $l'x + m'y = 1$ will meet in a point, provided $a^2ll' = b^2mm' = -1$; thus, if the equation to F_1F_2 be $\lambda x + \mu y = 1$, the two equations

$$\lambda x + \mu y = 1, \quad xx_1/a^2 + yy_1/b^2 = 1,$$

must satisfy the above conditions, *i.e.*, $\lambda x_1 = \mu y_1 = -1$; therefore the equation to F_1F_2 is $\quad x/x_1 + y/y_1 + 1 = 0.$

Hence the following construction for C.

Take P, the point on the ellipse diametrically opposite to P, and draw P′M, P′N perpendiculars on the axes. MN meets the ellipse in the points F_1, F_2, and the normals at F_1, F_2 intersect in C.



2. Put $\quad x/a = \xi$, $y/b = \eta$, $x_1/a = \xi'$, $y_1/b = \eta'$.

Then the equation to F_1F_2 is $\xi/\xi' + \eta/\eta' + 1 = 0$, subject to $\xi'^2 + \eta'^2 = 1$.

To find the envelope differentiate these equations, taking ξ_1, η_1 as variables, and use the method of undetermined multipliers, and we have

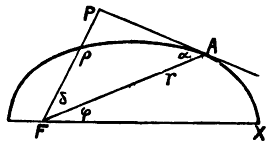
$$\xi/\xi'^2 = k\xi', \quad \eta/\eta'^2 = k\eta';$$

whence $k = -1$, and therefore $-\xi' = (\xi)^{\frac{1}{3}}$ and $-\eta' = (\eta)^{\frac{1}{3}}$; therefore the envelope is given by $\xi^{\frac{2}{3}} + \eta^{\frac{2}{3}} = 1$, *i.e.*, $(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1$.

1280. EDITOR.)—A given angle revolves round its vertex, which is fixed at the focus of a conic, and a tangent is drawn to the conic at

the point where it is cut by *one* of the sides of the angle; find the locus of the point in which this tangent meets *the other* side of the angle.

Solution by Professor LAMPE.



Let AFP $= \delta$ be the revolving angle, F the focus of the conic whose equation is $r = p/(1-\epsilon\cos\phi)$; then

$$\cot\alpha = -dr/rd\phi = \epsilon\sin\phi/(1-\epsilon\cos\phi).$$

Putting FP $= \rho$, we have at once

$$r/\rho = \sin(\alpha+\delta)/\sin\alpha$$
$$= \cos\delta + \cot\alpha\sin\delta.$$

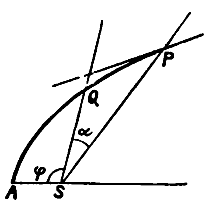
Solving for ρ and substituting for r and $\cot\alpha$ their values, we have

$$\rho = p/\{\cos\delta\,(1-\epsilon\cos\phi)+\epsilon\sin\phi\sin\delta\}.$$

Making $p/\cos\delta = p'$, $\epsilon/\cos\delta = \epsilon'$, $\phi+\delta = \psi$, we may write

$$\rho = p'/(1-\epsilon'\cos\psi).$$

This is the polar equation of the locus required; it is a conic section with F as focus, FX as axis, $2p'$ as latus rectum, and ϵ' as numerical eccentricity.



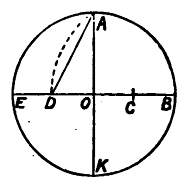
[Mr. TUCKER gives the solution thus:—

Let $\angle ASQ = \phi$, $\angle PSQ = \alpha$,

equation to conic $l = r\{1+e\cos(\theta-\alpha)\}$;

then equation to SQ is $\theta = \phi$, and to tangent at P is $l = r\{e\cos\theta+\cos(\theta-\phi-\alpha)\}$; hence the locus required is $l = r(e\cos\phi+\cos\alpha)$; that is, $l\sec\alpha = r(1+e\sec\alpha\cos\phi)$; that is to say, a confocal conic L.R $= l\sec\alpha$,

and eccentricity $= e\sec\alpha$.]



12251. (J. H. HOOKER, M.A.)—Prove that the side of a regular pentagon in a circle may be got by the following process:—EOB, AOK are diameters at right angles; C is the mid-point of OB; arc AD is drawn with centre C; line AD is side of pentagon. Show the correctness of the solution.

Solution by W. J. DOBBS, M.A.; Prof. MUKHOPADHYÂY; *and others.*

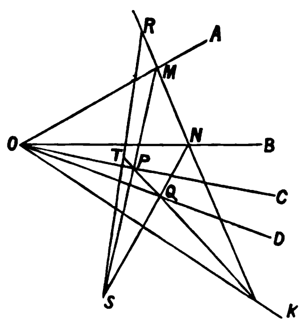
Let r be the radius of the circle, then OD is obviously equal to a side of a regular inscribed decagon; then

$$OD = 2r\sin 18^\circ,\quad AD^2 = r^2+4r^2\sin^2 18^\circ = 4r^2\sin^2 36^\circ;$$

$$\therefore\ AD = 2r\sin 36^\circ = \text{side of regular pentagon}.$$

12200. (Professor DUPORCQ.)—Étant donnés deux points R et S dans le plan de quatre droites concourantes OA, OB, OC, OD, par le point R on mène une transversale quelconque RMN qui coupe OA, OB aux points M et N. Les droites SM, SN rencontrent les droites OC, OD en P et Q. Démontrer que la droite PQ passe par un point fixe.

Solution by H. W. CURJEL, B.A.; W. J. DOBBS, B.A.; *and others.*



Consider OC, OD as the projections of OA, OB, S being centre of projection; then the axis of projection will be a straight line OK through O. Hence MN, PQ cut on OK, and PQ cuts SR in a point T, which is fixed, since it is the projection of R. If the lines of the figure are not coplanar, OS must evidently be the intersection of planes OAC, OBD, and R must be in the plane OAB.

1095. (EDITOR.)—A straight line intersects both loops of the lemniscate, one loop in P, Q, and the other in P′, Q′. Prove that the mid-points of PQ and P′Q′ are equidistant from the centre of the curve.

Solution by Professor LAMPE.

Let $(x^2+y^2)^2 = a^2(x^2-y^2)$ be the equation of the curve, $y = mx+n$ that of the intersecting straight line. Forming the biquadratic equation for the coordinates of the common points P, Q, P′, Q′, we get

$$x_1+x_2+x_3+x_4 = -mn/(1+m^2), \quad y_1+y_2+y_3+y_4 = n/(1+m^2).$$

Let (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) belong respectively to P, Q, P′, Q′; (x', y') to the mid-point M′ of P and Q, (x'', y'') to the mid-point M″ of P′ and Q′, (x, y) to the mid-point M of M′ and M″; then

$$x' = \tfrac{1}{2}(x_1+x_2), \quad y' = \tfrac{1}{2}(y_1+y_2), \quad x'' = \tfrac{1}{2}(x_3+x_4), \quad y'' = \tfrac{1}{2}(y_3+y_4),$$
$$x = \tfrac{1}{2}(x'+y'') = \tfrac{1}{4}(x_1+x_2+x_3+x_4) = -mn/\{4(1+m^2)\},$$
$$y = \tfrac{1}{2}(y'+y'') = \tfrac{1}{4}(y_1+y_2+y_3+y_4) = n/\{4(1+m^2)\}.$$

Hence $x = -x/m$, or the locus of the mid-point M between M′ and M″ for parallel lines $y = mx+n$ (or the centre of mean distances of P, Q, P′, Q′) is a straight line perpendicular to their common direction and passing through the origin O. But MM′ = MM″; therefore OM′ = OM″.

1. The theorem remains true, if we choose any two of the three couples which may be formed with the four intersections of the lemniscate and the straight line.

2. The demonstration is only based on the term $(x^2+y^2)^2$; consequently it holds for any curve $(x^2+y^2)^2 = a^2 . f(x, y)$, $f(x, y)$ being the general function of the second degree in x and y.

3. If m is constant in the equation $y = mx+n$, the locus of M (centre of mean distances for the four points of intersection) is shown to be a straight line perpendicular to the given direction. If the given line $y = mx+n$ passes through a given point C, the locus of M will consequently be a circle having CO as diameter.

[Mr. Curjel puts the solution thus:—

Take the equation to the lemniscate in the form $(x^2+y^2)^2-2c(x^2-y^2)=0$, and that to the straight line PQP′Q′, as $\dfrac{x-a}{\cos\alpha} = \dfrac{y-b}{\sin\alpha} = r$, with the condition $a\cos\alpha + b\sin\alpha = 0$; then (a, b) is the foot of the perpendicular from the centre O of the curve on PQP′Q′; and at the points of intersection

$$(r^2+2a^2)^2 - 2c^2\left\{r^2\cos 2\alpha + 2r(a\cos\alpha - b\sin\alpha)\right\} = 0\,;$$

hence the sum of the four values of r given by this equation $= 0$. Thus the mid-points of the straight line joining any pair of the points PQP′Q′, and of the straight line joining the other pair are equidistant from (a, b), and therefore from the centre O.]

6609. (J. Hammond, M.A.)—Find the approximate value of the following series, and its equivalent integral:—

$$\frac{1}{2!}+\frac{1}{3!}(1+\tfrac{1}{2})+\frac{1}{4!}(1+\tfrac{1}{2}+\tfrac{1}{3})+\frac{1}{5!}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})+\ldots \equiv \int_0^1 e^x \log\left(\frac{x}{1-x}\right)dx.$$

Solution by H. J. Woodall, A.R.C.S.

1. Dealing with the series first, we can find its numerical value to be

$$0{\cdot}84748,00638,72532,46456,\ldots.$$

2. We have
$$\int (e^x-1)\,x^{-1}\,dx = x+\tfrac{1}{2}x^2/2!+\tfrac{1}{3}x^3/3!+\ldots \quad\ldots\ldots\ldots\ldots(a),$$
$$x\int\left\{e^x-(1+x)\right\}x^{-2}\,dx = x^2/2!+\tfrac{1}{2}x^3/3!+\ldots \quad\ldots\ldots\ldots\ldots(b),$$
$$x^2\int\left\{e^x-(1+x+x^2/2)\right\}x^{-3}\,dx = x^3/3!+\ldots \quad\ldots\ldots\ldots\ldots(c),$$
&c.,

whence, taking limits 1 and 0, and adding (b), (c), &c., we get the series.

3. The given integral

$$\int_0^1 e^x\log\left(\frac{x}{1-x}\right)dx = \int_0^1 e^x\log x\,dx - \int_0^1 e^x\log(1-x)\,dx$$
$$= \int_0^1 e^x\log x\,dx - \int_0^1 e^{(1-x)}\log x\,dx = \int_0^1 (e^x - e\,.\,e^{-x})\log x\,dx.$$

This may be expanded by the aid of § 2391, CARR's *Synopsis of Elementary Results in Pure Mathematics.*

$$= \Sigma\left[-1/\{(p+1)(p+1)!\}\right] + e\Sigma\left[(-1)^p/\{(p+1)(p+1)!\}\right],$$

of which I have obtained the numerical value

$$\cdot 84748{,}00638{,}72532{,}46461.$$

We can say that to eighteen places of decimals the integral and series are equivalent. This is a kind of integral to which we cannot apply GAUSS's formula of integration by approximation.

TABLE OF VALUES OF

$$\Sigma_{k+1}^{\infty}\left(\frac{1}{n-k}\cdot\frac{1}{n!}\right) = \int_0^1 \left[e^x - (1 + x + \&\text{c.} + x^{k-1}/(k-1)!\right] x^{-k}\,dx.$$

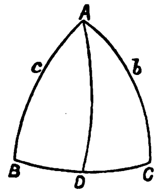
k					
0	1·31790	21514	54403	89486	00082
1	·59962	03229	95358	65949	97211
2	·19066	92472	68156	71206	97178
3	·04635	13618	25259	38112	53647
4	·00910	07166	74886	86977	46022
5	·00149	71109	76501	66021	61959
6	·00021	18063	45501	99673	52437
7	·00002	62780	20060	71647	76009
8		29024	25285	66529	83940
9		02888	37714	74108	85567
10		00261	52505	39854	46080
11		00021	71995	72984	94409
12		00001	66593	25486	96241
13			11869	77090	95300
14			00789	59167	58233
15			00049	25470	67335
16			00002	89240	03272
17				16044	44756
18				00843	29058
19				00042	11247
20				00002	00310
21					09095
22					00395
23					00016
Sum 1 to 23	·84748	00638	72532	46456	09717

For convenience I give here (from *Encyclopedia Britannica*, Art. "Logarithms")

$$e = 2{\cdot}71828{,}18284{,}59045{,}23536{,}02874.$$

12234. (Professor Macfarlane.) — Sin a sin B = sin b sin A is the spherical analogue of a sin B = b sin A; what is te spherical analogue of the complementary theorem $a \cos B + b \cos A = c$?

Solution by Beatrice A. Ward; Professor Mukhopadhyay; *and others.*



ABC is a spherical triangle. Draw AD at right angles to BC. In ΔADC, tan DC = tan b cos C. In ΔADB, tan DB = tan c cos B; therefore

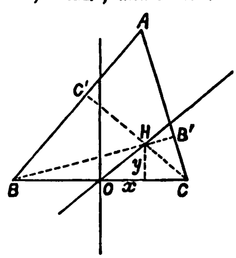
$$\tan a = \frac{\tan b \cos C + \tan c \cos B}{1 - \tan b \tan c \cos B \cos C},$$

which reduces to the formula

$$\cot b \sin a = \cos a \cos C + \sin C \cot B.$$

12231. (Professor Zerr.)—Given the base of a triangle and the difference of the tangents of the angles at the base; show that the locus of the point of intersection of the perpendiculars from the angles on the sides is a straight line through the centre of the base.

Solution by Professor Droz-Farny; R. F. Davis, M.A.; *and others.*



Soient BC la base fixe, O son point milieu, B′ et C′ les pieds des hauteurs abaissées de B et C, H l'orthocentre. On a

$$\tan B = \tan C'CB = (\tfrac{1}{2}a - x)/y,$$
$$\tan C = \tan B'BC = (\tfrac{1}{2}a + x)/y;$$

donc $m = 2x/y$, ou $y = 2x/m$, ligne droite passant par l'origine et faisant avec la base un angle ϕ tel que $\tan \phi = 2/m$.

Dans les mêmes conditions le lieu de A sera l'hyperbole $mx^2 + 2xy = \tfrac{1}{2}ma^2$.

1136. (Editor.)—Two tangents to a semi-cubical parabola include a given angle; find (1) the locus of their point of intersection; and (2) what the locus becomes when the given angle is a right angle.

Solution by R. Tucker, M.A., *and* Professor Lampe.

Suppose the curve given by the equations

$$x = 4am^2/9, \quad y = 8am^3/27, \quad m = 3y/2x;$$

then tangents are $y - mx = -4am^3/27$, $y - m'x = -4am'^3/27$, with $m - m' = c(1 + mm')$, where c is a constant.

The locus curve is given by

$$27x = 4a\,(m^2+mm'+m'^2), \quad 27y = 4amm'\,(m+m').$$

Write $\qquad 4a\lambda = 27x, \quad 4a\mu = 27y, \quad mm' = z$;

then we have to eliminate z from

$$z^3+\lambda z^2-\mu^2 = 0, \quad c^2z^4+2\,(c^2+2)\,z^3+c^2z^2-\mu^2 = 0\,;$$

i.e., from $\qquad c^2z^2+(2c^2+3)\,z+c^2-\lambda = 0,$

and $\qquad (c^2-\lambda)\,z^2+(c^2\lambda+c^2\mu^2-\lambda^2)\,z+\mu^2\,(2c^2+3) = 0.$

The eliminant is $\;(pq\mu^2-r^2)^2+(qr-ps)(q^2\mu^2-rs) = 0,$

where $\quad p = c^2, \; q = 2c^2+3, \; r = c^2-\lambda, \; s = \lambda\,(c^2-\lambda)+c^2\mu^2.$

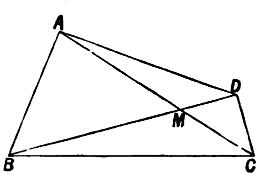
The curve, in general, is of the fifth degree.

If $mm' = \pm 1$, $\mu^2 = \lambda \pm 1$, *i.e.*, in each case a parabola.

12037. (Professor Delahaye.)—Si M est le point de rencontre des diagonales AC, BD d'un quadrilatère ABCD, on a la relation

$$AC\,.\,BD\,(AM-MC) = \{(AB^2-BC^2)\,MD+(AD^2-DC^2)\,MB\}.$$

Solution by H. W. Pyddoke; T. Savage; *and others.*



We have $AB^2\,.\,MD+AD^2\,.\,MB$

$= BM^2\,.\,MD+BM\,.\,MD^2+BD\,.\,AM^2$;

similarly $BC^2\,.\,MD+DC^2\,.\,MB$

$= BM^2\,.\,MD+BM\,.\,MD^2+BD\,.\,MC^2$;

hence, subtracting, we have

$(AB^2-BC^2)\,MD+(AD^2-DC^2)\,MB$

$= BD\,(AM^2-MC^2) = BD\,.\,AC\,(AM-MC).$

12226. (J. D. H. Dickson, M.A.)—If

$$\{\sin(\alpha-\beta)+\cos(\alpha+2\beta)\sin\beta\}^2 = 4\cos\alpha\sin\beta\sin(\alpha+\beta),$$

and α, β are each less than a right angle, prove that

$$\tan\alpha = \tan\beta\,\{(\sqrt{2}\,.\,\cos\beta-1)^{-2}-1\}.$$

Solution by the Proposer.

As we have to find $\tan\alpha$, arrange the given equation with regard to $\sin\alpha$ and $\cos\alpha$. Thus, squaring and expanding the right-hand of the given equations, and transposing the left-hand, we get

$$\tan^2\alpha\cos^2 2\beta-2\tan\alpha\;2\sin\beta\cos\beta\,(2-\cos 2\beta)-2\sin^2\beta\,(3-\cos 2\beta) = 0,$$

whence $\cos^2 2\beta \tan\alpha = 2\sin\beta\cos\beta\,(2-\cos 2\beta) \pm 2\sqrt{2}\sin\beta$;

$\therefore\ \tan\alpha/\tan\beta = \left\{2\cos^2\beta\,(3-2\cos^2\beta) \pm 2\sqrt{2}\cos\beta\right\}/(2\cos^2\beta-1)^2$;

putting $k^2 = 2\cos^2\beta$, this becomes

$$\tan\alpha/\tan\beta = -(k^2-2k)/(k-1)^2, \text{ or } -(k^2+2k)/(k+1)^2,$$
$$= (k-1)^{-2}-1, \text{ or } (k+1)^{-2}-1;$$

hence the theorem, if α and β are both to be less than a right angle.

12241. (J. Brill, M.A.)—If $\xi = \alpha$, $\eta = \beta$ be a particular solution of the equations $m\dfrac{\partial\xi}{\partial x} = \dfrac{\partial\eta}{\partial y}$, $m\dfrac{\partial\xi}{\partial y} = -\dfrac{\partial\eta}{\partial x}$, where m is a specified function of x and y, prove that we may write $\xi = \dfrac{\partial\chi}{\partial\beta}$, $\eta = \dfrac{\partial\chi}{\partial\alpha}$, where χ satisfies the equation $\dfrac{\partial^2\chi}{\partial\alpha^2} + m^2\dfrac{\partial^2\chi}{\partial\beta^2} = 0$.

Solution by the Proposer; Professor Bhattacharya; *and others.*

The given equations express the conditions that the value of each of the expressions $\dfrac{md\xi + id\eta}{dx+idy}$, $\dfrac{md\alpha + id\beta}{dx+idy}$ is independent of the ratio $dx : dy$; hence the value of $\dfrac{md\xi + id\eta}{md\alpha + id\beta}$ is independent of the ratio $d\alpha : d\beta$. The conditions that this may be the case are $\dfrac{\partial\xi}{\partial\alpha} = \dfrac{\partial\eta}{\partial\beta}$, $m^2\dfrac{\partial\xi}{\partial\beta} = -\dfrac{\partial\eta}{\partial\alpha}$, of which equations, the first indicates the existence of a function χ, such that $\xi = \dfrac{\partial\chi}{\partial\beta}$, $\eta = \dfrac{\partial\chi}{\partial\alpha}$, and, substituting these values in the second equation, we obtain

$$\frac{\partial^2\chi}{\partial\alpha^2} + m^2\frac{\partial^2\chi}{\partial\beta^2} = 0.$$

12187. (Professor Hudson, M.A.)—If a jot be the time that light takes to advance a tenth of a millimetre, and if there be a thousand million molecules in a cubic micron of air, a micron being the thousandth of a millimetre, and if each molecule encounters another every 420 jots and moves in a straight line ·07 micron long between the encounters, find to the nearest unit how many thousand centuries it would take light to traverse a distance equal to the aggregate of all the paths between the encounters that occur in a cubic centimetre of air in a second.

Solution by H. J. Woodall, A.R.C.S.

The number of molecules in a cubic centimetre of air $= 10^{21}$.

Number of times in one second that a certain molecule encounters another $= A/4 \cdot 2$, where A is the velocity of light in centimetres per second.

Between two encounters, each molecule traverses a path $= \cdot 07 \times 10^{-4}$ cm.;

$$\therefore \quad \cdot 07 \times 10^{-4} \times A/4\cdot 2 = \tfrac{1}{6}10^{-5}A \text{ per second};$$

hence the sum of all the paths in one second

$$= \tfrac{1}{6}A \,.\, 10^{-5} \times 10^{21} = \tfrac{1}{6}A \,.\, 10^{16}.$$

Light would travel this distance in

$$\tfrac{1}{6}A \times 10^{16} / \left\{10^5 \times 365\cdot 25 \times 24 \times 60 \times 60A\right\} = 528 \text{ thousand centuries.}$$

[The PROPOSER's solution is as follows:— 1 cub. cm. $= 10^{12}$ cubic microns, and contains 10^{21} molecules; in one jot each molecule travels 1/420 of ·07 of ·001 mm.; this distance is travelled by light in 1/420 of ·07 of ·01 jot; therefore the distance travelled by 1 molecule in 1 sec. is travelled by light in 1/420 of ·07 of ·01 sec. = 1/600000 sec.; therefore the time light takes to travel a distance equal to the aggregate of the paths of all the molecules in 1 sec. is $10^{21}/600000$ sec.,

$$= 10^{16}/(6 \times 60 \times 60 \times 24 \times 365 \times 100000) \text{ thousand centuries,}$$
$$= 528 \text{ thousand centuries approximately.]}$$

12208. (J. H. HOOKER, M.A.)—A perfect number is defined by an old arithmetician as one which is equal to the sum of its divisors, excluding itself and including unity. One is 28, which equals the sum of 1, 2, 4, 7, 14; find others.

Solution by R. CHARTRES; Professor CHAKRIVARTI; *and others.*

Take any power of 2, say 16, such that its double diminished by unity is a prime number;

$\therefore$ $16 \times 31 \equiv (1+2+4+8+16)+31(1+2+4+8) =$ a perfect number;

$\therefore$ $2^{n-1}(2^n-1)$ will give perfect numbers provided (2^n-1) is a prime.

If $n = 2, 3, 5, 7, 13$, &c., we get 6, 28, 496, 8128, 33550336, &c.

9636. (CHARLES L. DODGSON, M.A.)—If three numbers, not in Arithmetical Progression, be such that their sum is a multiple of 3, prove that the sum of their squares is also the sum of another set of 3 squares, the two sets having no common term.

Solution by W. S. FOSTER, Professor NASH, *and others.*

If $a+b+c = 3m$, then $a^2+b^2+c^2 = (2m-a)^2+(2m-b)^2+(2m-c)^2$, and $2m-a$ is not equal to a or b or c unless these quantities are in Arithmetical Progression.

12112. (W. J. GREENSTREET, B.A.)—Prove that, whatever the law of force by which planets are retained in their circular orbits, the angle subtended at the Sun by two planets mutually stationary is

$$\cos^{-1}(au+bv)/(av+bu),$$

u, v being velocities of planets, a, b radii of their orbits. Show also, in the stationary position, they really recede from one another with a velocity $$(v^2-u^2)^{\frac{1}{2}}\left\{(av-bu)^{\frac{1}{2}}/(av+bu)^{\frac{1}{2}}\right\}.$$

Solution by the PROPOSER.

Let the law be $\alpha = \lambda a^n$, $\beta = \lambda b^n$, where a, b are the circular radii and α, β are angular velocities. Then, if ϕ be the difference of longitudes,

$$\cos\phi = (a^2\alpha + b^2\beta)/\left\{ab\,(\alpha+\beta)\right\} = (a^{n+2}+b^{n+2})/\left\{a^{n+1}b + b^{n+1}a\right\}.$$

The actual velocities u and v are given by

$$u = a\alpha = \lambda a^{n+1}; \quad v = b\beta = \lambda b^{n+1}; \quad \therefore \cos\phi = (au+bv)/(av+bu).$$

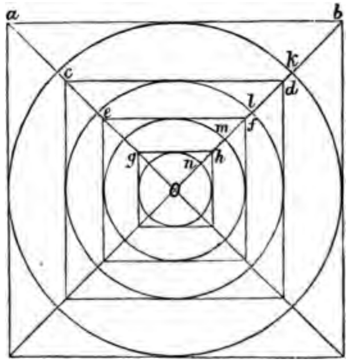
Resolving the relative velocities, we get

$$a\alpha\cos\alpha t + b\beta\cos\beta t \quad\text{and}\quad a\alpha\sin\alpha t - b\beta\sin\beta t;$$

$$\begin{aligned}\therefore\quad (\text{actual velocity})^2 &= a^2\alpha^2 + b^2\beta^2 - 2ab\,\alpha\beta\cos(\alpha-\beta)\,t \\ &= v^2+u^2 - 2ab\,\alpha\beta\,(a^2\alpha+b^2\beta)/\left\{ab\,(\alpha+\beta)\right\} \\ &= v^2+u^2 - 2\alpha\beta\,(\alpha a^2+\beta b^2)/(\alpha+\beta) \\ &= v^2+u^2 - 2uv\,(au+bv)/(bu+av).\end{aligned}$$

11939. (Professor SHIELDS.)—A surveyor owned, and had his residence H in the centre of, a large square tract of land A, whose perimeter contained as many inches as there were acres in its area, in which was inscribed the largest circle B possible. The circumference of the circle contains as many inches as there are acres in its area. Within the circle B is another similar square tract C, and inscribed circle D. The perimeter of the square tract C contains as many yards as there are acres in its area, and the circumference of the inscribed circle D contains as many yards as there are acres in its area; and in the circle D is laid off another similar square tract E, and inscribed circle F. The perimeter of the square tract E contains as many rods as there are acres in its area; and the inscribed circle F contains as many rods as there are acres in its area; and in the circle F there is a square house-lot G, on which stands the largest round-based house H possible; the perimeter of the square house-lot G contains as many feet as there are square yards in its area; and the circumference of the round-based house H contains as many feet as there are square yards covered by the house. Give the perimeter, circumference, and area of each tract of land.

Solution by T. SAVAGE; Professor CHARKRIVARTI; *and others.*



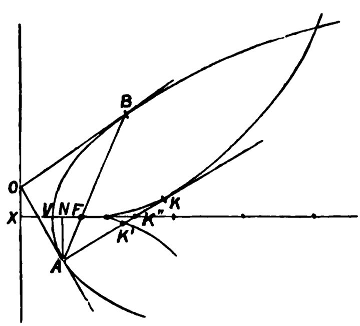
Suppose the circumference of H contains x feet. A side of the escribed square G will therefore measure x/π feet. The area of circle H is $x^2/4\pi$ square feet. But circle H is to contain x square yards. Therefore $x^2/36\pi = x$. Hence the circumference of H is 36π feet, its area is 36π square yards, the perimeter of G is 144 feet, and its area is 144 square yards.

In a similar manner we obtain the areas of F and E to be 640π acres and 2560 acres respectively, the areas of D and C to be 19360π acres and 77440 acres respectively, and the areas of B and A to be $19360 \times (36)^2\,\pi$ acres and $77440 \times (36)^2$ acres respectively.

12207. (R. H. W. WHAPHAM, B.A.)—From any point O on the directrix of a parabola, of vertex V and focus F, two tangents OA, OB are drawn to it. The normal at A to parabola meets evolute at K′, touches it at K, and meets axis in K″; prove that

$$(1)\ KK'' : KA = OA^2 : AB^2; \quad (2)\ OA \, . \, OB / AB^3 = VF.$$

Solution by R. TUCKER, M.A.; *the* PROPOSER; *and others.*



1. Let AN be the ordinate of A; then

$$4a^2 \, . \, AK = AK''^3,$$

and $AK''^2 = 4a \, . \, FA$;

$\therefore\ KK'' \, . \, 4a^2 = AK'' \, . \, AN^2$,

and $KK'' : AK = AN^2 : AK''^2$

$= OA^2 : AB^2$

(because the triangles OFA, ANK″ are similar).

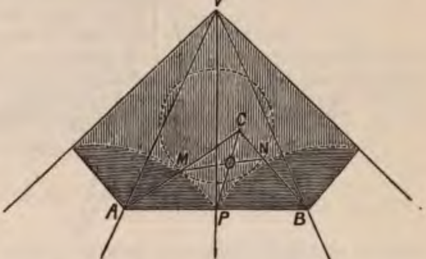
2. $OA \, . \, OB = 2\Delta AOB$

$= OF \, . \, AB$,

$$\therefore\quad OA^2 \, . \, OB^2/AB^3 = OF^2/AB = AF \, . \, FB/AB = VF = \text{constant}.$$

12240. (J. W. RUSSELL, M.A.)—Two equal right circular cones, which have their vertices coincident and their axes horizontal, touch along a common generator. A sphere moves along the cones under the action of gravity from a given position to the position in which the normals of the cones at the points of contact are equal to the radius of the sphere. Show that the final velocities of the centre of the sphere in the cases when the sphere is (1) perfectly rough, (2) perfectly smooth, are in the ratio $(15)^{\frac{1}{2}} : (23)^{\frac{1}{2}}$.

Solution by the PROPOSER; Professor ANDERSON, M.A.; *and others.*



The equations of motion are

$$\tfrac{1}{2}mk^2\omega^2 + \tfrac{1}{2}mv_1^2 = mg\,(y_0 - y_1),$$

$$\tfrac{1}{2}mv_2^2 = mg\,(y_0 - y_1);$$

$$\therefore\quad k^2\omega^2 + v_1^2 = v_2^2$$

and $v_1 = p\omega$;

$$\therefore\quad v_1^2 : v_2^2 = p^2 : p^2 + k^2,$$

where $k^2 = \frac{2}{5}a^2$.

Let C be the centre of the sphere, M and N the points of contact. Let the normals CM and CN cut the axes in A and B. Join AB. Then AMCNBP is a plane.

Now MN is instantaneous axis, and $p = \text{CO}$. Also $\text{PA} = \text{AM}$, being normals of the cone $= a = \text{CM}$; $\therefore\ \text{CO}^2 = \frac{1}{4}\text{CP}^2 = \frac{1}{4}(4a^2 - a^2) = \frac{3}{4}a^2$;

$$\therefore\quad v_1^2 : v_2^2 = \tfrac{3}{4}a^2 : \tfrac{3}{4}a^2 + \tfrac{2}{5}a^2 = 15 : 23.$$

7600. (Professor HAUGHTON, F.R.S.)—Find the TARTINI tones of the following combinations of notes sounded together:—c and g, c' and f', c' and a, c' and e.

Solution by H. J. WOODALL, A.R.C.S.

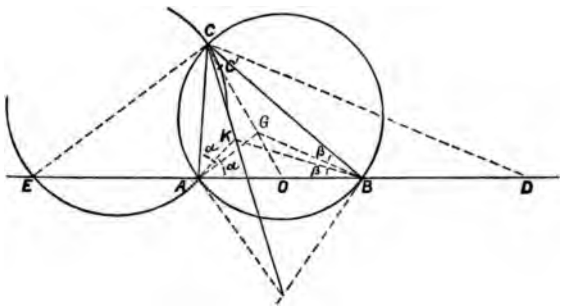
The TARTINI tone of a combination is the highest common sub-harmonic of the combination. The numeric of its frequency is the G.C.M. of the numerics of the frequencies of the tones of the combination.

The frequency ratio of c and g is $1 : \frac{3}{2} = 2 : 3$; therefore the TARTINI tone is $\text{C} = \frac{1}{2}c$. So of c' and f', c' and a, c' and e, the TARTINI tones are F, F_1, and C_1, respectively.

12223. (Professor SCHOUTE.)—Two vertices of a triangle being given in position, examine the correspondence between the third vertex and the Lemoine-point of the triangle.

Solution by Professors DROZ-FARNY, MOREL, *and others.*

Soient A et B les sommets donnés; à chaque position du troisième sommet C correspond une position unique du point de Lemoine K. Ce



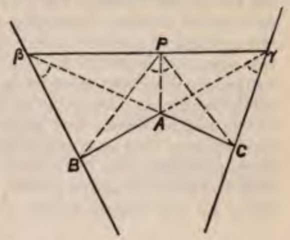
point est en effet le centre d'homologie du triangle ABC et du triangle dont les côtés sont tangents en A, B, C à la circonférence circonscrite. Par contre à chaque position donnée de K correspondent deux positions du sommet C. En effet: on sait que le centre de gravité G et le point de Lemoine sont conjugués isogonaux. Posons angle KBA $= \beta$ et KAB $= \alpha$, on aura donc aussi angle GBC $= \beta$ et GAC $= \alpha$. Les parallèles menées de C à GB et GA coupent en D et E la base AB et on aura O étant le

milieu de AB, $\quad OG : OC = OB : OD = OA : OE = 1 : 3,$

d'où $BD = BA = AE = c$. Les circonférences construites sur BD et AE et capables respectivement des angles β et α se couperont aux sommets cherchés C et C′. Comme $OB \cdot OD = OA \cdot OE = \frac{3}{4}C^2$ la droite CC′ passe par le milieu de AB et les points C et C′ sont conjugués par rapport à la circonférence de centre O et de rayon $\frac{1}{2}C\sqrt{3}$. Pour que les points C et C′ soient réels distincts ou coincidents il faut que le point K se trouve à l'intérieur de l'ellipse ou sur l'ellipse ayant A et B comme foyers et dont les axes auraient respectivement pour longueurs $\frac{2}{3}\sqrt{3}C$ et $\frac{1}{3}C\sqrt{3}$.

12249. (J. GRIFFITHS, M.A.)—Prove that the locus of a point P at which the two sides AB, AC of a triangle ABC subtend equal angles is a circular cubic having A for a double point. Trace the curve.

Solution by Professors DROZ-FARNY, ZERR, *and others.*



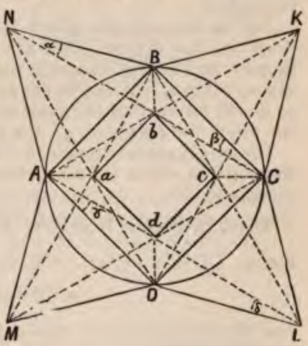
Soit P un point du lieu. La perpendiculaire en P sur PA rencontre en β et γ les perpendiculaires en B et C respectivement sur AB et AC. Dans le quadrilatère $BAP\beta$ on a angle $BPA = B\beta A$, de même dans le quadrilatère $CAP\gamma$ on a angle $APC = A\gamma C$; et comme $BPA = APC$ il en résulte que les triangles $B\beta A$ et $C\gamma A$ sont semblables. Donc $B\beta/C\gamma = BA/CA$.

La droite $\beta\gamma$ enveloppe donc une parabole, et comme les bissectrices de l'angle A sont deux cas particuliers de la tangente $\beta\gamma$, A est un point de la directrice de cette parabole.

Le lieu cherché est donc une podaire de parabole par rapport à un point de sa directrice, soit une focale à nœud de QUÉTELET. C'est une strophoïde oblique, cubique circulaire ayant le point A comme point double avec les bissectrices de l'angle A comme tangentes et passant par B et C. Le lieu complet se compose de deux cubiques analogues.

12168. (R. TUCKER, M.A.)—ABCD is a quadrilateral in a circle; K, L, M, N are the vertices of equilateral triangles described externally on BC, CD, DA, AB respectively. BM, DN cut in a; AK, CN in b; DK, BL in c; and AL, CM in d; prove that $abcd$ is circumscriptible.

Solution by Professors DROZ-FARNY, LAMPE, *and others.*



Les triangles NAD et BAM sont égaux. Comme les côtés égaux AN et AB, AD et AM sont inclinés de 60° l'un sur l'autre, il en sera de même des côtés égaux BM et DN. Le quadrilatère $ANBa$ est donc inscriptible. Il en résulte que les points a, b, c, et d ne sont rien d'autres que les points d'intersection consécutifs des circonférences circonscrites aux quatre triangles équilatéraux. Comme

$\sphericalangle\, baB = bNB = \alpha,\quad daD = dAD = \gamma;$

$$\sphericalangle\, a = 120 - (\alpha + \gamma);$$

de même $\sphericalangle\, c = 120 - (\beta + \delta)$.

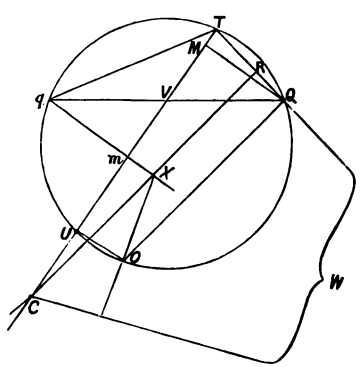
Donc $a + c = 240 - (\alpha + \beta + \gamma + \delta)$.

Or $\alpha + \beta = 120 - B$, $\gamma + \delta = 120 - D$, $\alpha + \beta + \gamma + \delta = 240 - B - D = 60°$.

Donc $a + c = 180°$. Le quadrilatère $abcd$ est donc *inscriptible*.

12254. (F. S. Macaulay, M.A.)—Prove the following construction for the circle through the feet of the three normals from any given point on a given normal chord of a given conic. Let O be the given point on the normal at a given point Q, let QQ′ be the diameter through Q, let the diameter parallel to OQ and the perpendicular through O to the conjugate diameter meet in X; then the circle on XQ′ as diameter is the circle required. Deduce the construction for the circle through the feet of the normals from any given point to a given parabola. Also deduce the theorem that any circle through a point on a conic and the foot of the perpendicular from the centre of the conic to the tangent at the point cuts the conic in three points, the normals at which are concurrent. The circle on any central radius as diameter is a particular case of this.

Solution by R. F. Davis, M.A.; *the* Proposer; *and others.*



Let TQ, Tq be two tangents to an ellipse centre C, CT bisecting Qq in V; O the extremity of the diameter through T of the circumcircle of the triangle TQq, so that QO, qO are normals to the ellipse at Q, q. Draw qm perpendicular to CT meeting CR, the perpendicular from C on QT, in X.

Then, if Q′ (not shown in the figure) be the other extremity of the diameter CQ, Q′q will be parallel to CV and the circle upon Q′X as diameter will pass through q.

Let us suppose the point Q fixed on the ellipse (with the tangent and normal thereat), and the point O upon the normal QO also given. Then, if it can be shown that the position of X is independent of q, it will follow that the same circle upon Q′X as diameter passes through the other two points q', q'', the normals at which pass through O.

Draw QM perpendicular to CT so that VM = Vm; and join O to U, the point in which CT meets the circle in figure, so that OU is perpendicular to CT.

Then, by similar triangles, CR.CX = Cm.CT, CR.OQ = UM.CT;

$$\text{hence}\quad CR\,(CX+OQ) = CT\,(Cm+UM) = CT\,(2CV-CM+UM)$$
$$= 2CT\,.\,CV - CU\,.\,CT = a^2+b^2$$

(see Milne and Davis's *Conics*, Part II., p. 163), or CX + OQ is constant, and X is fixed.

If CW be drawn perpendicular to OX to meet TQ in W,

$$RW : CR = OQ - RX : RQ,$$

or $$RW \,.\, RQ = CR\,(OQ - RX) = CR\,(OQ + CX) - CR^2$$
$$= a^2 + b^2 - CR^2 = CE^2 + RQ^2,$$

and $RQ \,.\, QW = CE^2$, CE being conjugate to CQ.

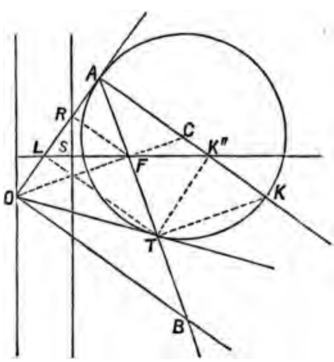
Thus CR, CW are conjugate diameters.

If RC be produced through C to R′, so that CR′ = CR, R′ is the foot of the perpendicular from C upon the tangent at Q′. The circle upon Q′X as diameter thus obviously passes through R′.

In the case of the parabola, Q′ may be taken to be the vertex A, and X is then the point whose abscissa exceeds that of O by 2AS, and whose ordinate is half that of O.

12243. (R. H. W. Whapham, B.A.)—From any point O on the directrix of a parabola, two tangents OA, OB are drawn to it. OA meets the axis in L, and LT is drawn parallel to OB to meet AB in T. Prove that the circle which touches parabola at A, and which passes through T, will also pass through K (see Quest. 12207) and will touch OT at T.

Solution by Professor Droz-Farny; *the* Proposer; *and others.*



Soient S le sommet de la parabole et F le foyer. Les deux tangentes OA et OB sont perpendiculaires l'une sur l'autre et OF perpendiculaire sur AB. La tangente OA coupe la tangente au sommet en R; on sait que angle

$$ARF = 90^\circ = ALT$$

et que AR = RL, donc AF = FT. Comme angle FAK = AK″F, AF = FK″ = FT; donc TK″ est perpendiculaire sur la normale en K″; et d'après un théorème connu, si TK est élevé en T perpendiculairement sur le rayon vecteur FA, K est le centre de courbure de A. OF coupe en C la normale; comme FC ∥ TK on a aussi AC = CK. La circonférence décrite de C comme centre avec CA comme rayon est donc tangente en A à la parabole; elle passe par TA par K, et comme OA = OT elle est tangente en T à OT.

12202. (R. Tucker, M.A.)—H, O, I are the orthocentre, circumcentre, and in-centre, respectively, of the triangle ABC. Hα is drawn parallel (1) to AO, (2) to AI, to meet BC in α. β, γ are analogous points on the other sides. Prove that, in both cases, Aα, Bβ, Cγ cointersect in a point.

Solution by Professors DROZ-FARNY, MOREL, *and others.*

1. Soit A′ le point milieu de BC, et O′ le symétrique de O par rapport à BC. Comme AH = 2OA′ = OO′ la droite HαO′ est parallèle à AO. Or triangle OαO′ est isoscèle, donc angle OαC = HαB, et comme H et O

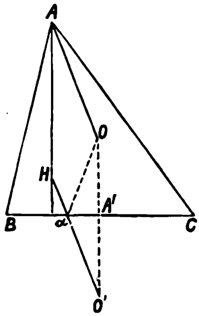


Fig. 1.

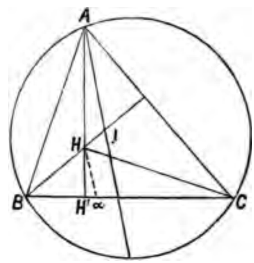


Fig. 2.

sont conjugués isogonaux, α sera le point de contact avec BC d'une conique inscrite dans le triangle et admettant H et O comme foyers. D'après PASCAL Aα, Bβ, Cγ concourent en un même point, *i.e.*,

$$a^2\alpha \cos A = b^2\beta \cos B = c^2\gamma \cos C.$$

2. Soit H′ le pied de la hauteur AH. Angle αHH′ = IAH = $\frac{1}{2}$(B − C). Comme angle BHH′ = C on aura BHα = $\frac{1}{2}$(B + C). Mais BHC = B + C, il en résulte que Hα est la bissectrice de l'angle BHC. On aura donc

$$B\alpha : C\alpha = BH : CH, \qquad C\beta : A\beta = CH : AH, \qquad A\gamma : B\gamma = AH : BH;$$

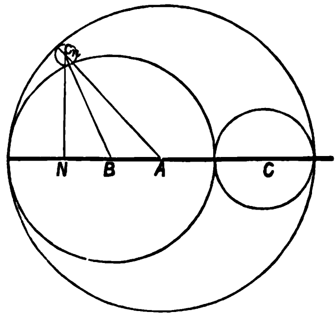
d'où $$B\alpha \,.\, C\beta \,.\, A\gamma = C\alpha \,.\, A\beta \,.\, B\gamma.$$

Les trois droites Aα, Bβ, Cγ concourent donc en un même point, *i.e.*,

$$a\alpha \cos A = b\beta \cos B = c\gamma \cos C.$$

7905. (R. LACHLAN, B.A.)—Three circles A, B, C are described having their centres in the same straight line; B and C touching one another externally, and touching A internally, and C_1 is a circle which touches A internally, and B and C externally, C_2 a circle touching A internally, and B and C_1 externally; and so on. Show that radius of $C_n = \dfrac{abc}{ab + n^2 \,.\, c^2}$, a, b, c being the radii of A, B, C.

Solution by H. J. WOODALL, A.R.C.S.



CASEY, *Sequel*, Book VI., Section IV., Cor. to Prop. 9, proves that the perpendicular from the centre of C_n on to the common diameter of circles AB and C is equal to $2nr$ where r is the radius of C_n.

Let A, B, C, C_n be the centres of the circles respectively; C_nN the perpendicular on ABC from C_n. Join C_nA, C_nB.

Then, by reducing the identity

$$NB + BA = NA,$$

we have

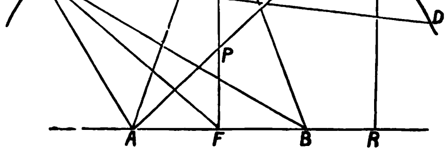
$$r\left\{n^2(a-b)^2 + ab\right\} = ab(a-b);$$

but $\quad a - b = c;$

$$\therefore \quad r = abc/(ab + n^2c^2).$$

12076. (I. ARNOLD.)—In a straight line given in position, find a point from which, if two straight lines be drawn to two given points, the sum of their squares shall be equal to a given square.

Solution by the PROPOSER.



Let A and B be the given points. Join them and bisect AB by the perpendicular FG. In FG take FP = FA, and draw APQ equal to the side of the given square. From Q let fall the perpendicular QR; then, from A as centre, with a radius equal to AR, describe a circle cutting FG in H; and, from F as centre, with FH as radius, describe another circle cutting the given line CD in C.

Then AC and BC are the lines required.

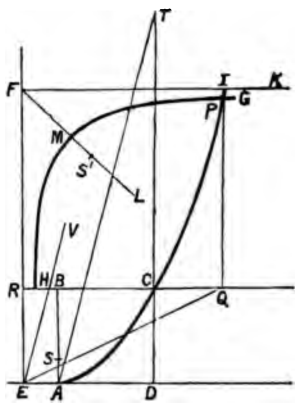
$$AC^2 + CB^2 = 2AF^2 + 2FH^2.$$

But $\quad AH^2 = AR^2 = AF^2 + FH^2;$

$$\therefore \quad 2AF^2 + 2FH^2 = 2AR^2 = AQ^2; \quad \therefore \quad \&c.$$

12247. (D. Biddle.)—Show that it is possible to trisect an angle by aid of the parabola and hyperbola in conjunction, after the manner set forth in Quest. 12169, and in the note on cubic equations appended to the solution of it.

Solution by the Proposer.



By a well-known formula,

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

Let $\tan 3A = a$, and $\tan A = x$. Then we have $a = (3x - x^3)/(1 - 3x^2)$, and $x^3 - 3ax^2 - 3x + a = 0$.

In a Graphic Method of Solving Cubic Equations (Vol. LXI., pp. 27, 28), it has been shown how the typical equation $\quad x^3 + px^2 + qx + r = 0$ might be transformed into

$$\frac{q^2 - 4pr}{4r^2} - \frac{x}{r} = \left(\frac{1}{x} + \frac{q}{2r}\right)^2;$$

and, in the present instance, this becomes $\dfrac{9 + 12a^2}{4a^2} - \dfrac{x}{a} = \left(\dfrac{1}{x} - \dfrac{3}{2a}\right)^2$, or $y = z^2$.

Hence $\qquad \dfrac{x}{a} = \dfrac{9 + 12a^2}{4a^2} - y$, and $\dfrac{1}{x} = z + \dfrac{3}{2a}$.

Moreover, $\qquad x/a \, . \, 1/x = 1/a = \frac{1}{4}(2/\sqrt{a})^2$.

Let ABCD be a square of side = unity, and let DAT = the angle to be trisected. Produce DC to cut AT in T; then DT = tan DAT = a. Produce DA to E, making AE = $3/(2a)$; also draw EF at right angles, making EF = $(9 + 12a^2)/(4a^2)$; and, drawing FK parallel to ED, bisect the angle EFK by FL. With focus S on AB, at $\frac{1}{4}$AB from A, describe the parabola ACI; and with focus S′ on FL, at $2/\sqrt{a}$ from F, describe the hyperbola GMH, having FE, FK as its asymptotes. From P, the point of intersection of the two curves, draw PQ perpendicular to BC produced, and join EQ. Then ∠ DEQ = $\frac{1}{3}$DAT, RQ being its co-tangent, = $1/x$. There is no need of proof of this if the propositions contained in Question 12169, and in the method of solving cubic equations above referred to, be accepted. [See Vol. LXI., pp. 27, 28, 58.]

12271. (Professor Hudson, M.A.)—Two watches are set together at noon; one gains 1 min. per week, the other loses 1 min. per week. How

soon (weeks, days, hours, minutes) will the hour and minute hands be in diametrically opposite directions for both watches at once?

Solution by D. Biddle.

Giving a general form to the answer, it may be stated that there are 11 positions on each dial at which diametrical opposition of the hands occurs, namely—$12\frac{6}{11}$, $1\frac{7}{11}$, $2\frac{8}{11}$, $3\frac{9}{11}$, $4\frac{10}{11}$, 6, $7\frac{1}{11}$, $8\frac{2}{11}$, $9\frac{3}{11}$, $10\frac{4}{11}$, $11\frac{5}{11}$; or at intervals of $1\frac{1}{11}$ of an hour. The positions for conjunction of the hands are also 11 in number, and midway between the others. These positions are entirely due to the mechanical arrangement by which the minute-hand travels 12 times as fast as the hour-hand, and they are in no way interfered with by the varying speed of the watch.

Let a, b = the respective gain and loss (in minutes per week) of the two watches A and B, and t = the position of the hour-hand of the true time-keeper T. Let k = one of the primary divisions or intervals above given, = $\frac{12}{11}$ of an hour. Then $\frac{60k}{a+b}$ = the time (in weeks) taken to separate the hour-hands of A and B by a distance = k. Let m = the total number of such units of separation accomplished before simultaneous opposition is attained. The proportions contributed by A and B, respectively, will be as $a : b$, namely, $\frac{ma}{a+b}$ and $\frac{mb}{a+b}$, and these give the relative position of t on T. The time (in weeks), given above, for the unit of separation, $= \frac{60k}{a+b} = \frac{10996\frac{4}{11}}{a+b}$ hours, making $\frac{10080k}{a+b}$. Let r (= a fraction of k) represent the distance, beyond one of the primary positions, of the starting-point of the two watches. Then $m\left(\frac{10080+a}{a+b}\right)+r$ and $m\left(\frac{10080-b}{a+b}\right)+r$ must be integral (and if one is, both will be). The number of different starts that can be made, between any two of the eleven primary positions, is determined by the denominator of the fractional part of $\frac{10080+a}{a+b}$, when reduced to its lowest form; the denominator itself represents the said number, and this must be multiplied by 11 to give the total number of possible starts on the dial as a whole. The respective times taken are given by $\frac{m\,(10996\frac{4}{11})}{a+b}$ in hours, by $\frac{m\,(458\frac{2}{11})}{a+b}$ in days, and by $\frac{m\,(65\frac{5}{11})}{a+b}$ in weeks.

In the case before us, $m\left(\frac{10080+a}{a+b}\right)+r$ becomes $m\left(\frac{10080+1}{2}\right)+r$, and, since $r=\frac{1}{2}$, it is clear $m=1$. Therefore the required time = 32 weeks, 5 days, 2 hours, 10 minutes, $54\frac{6}{11}$ seconds; so that t on T is at $2\frac{2}{11}$ hours past noon, or midway between two of the primary positions, as it was at the start. But the denominator of the fraction being 2, we know there is another starting-point within the compass of

each primary division. This we find by taking $m = 2$ and $r = 0$. Thus, if the two watches start at six o'clock, when their hands are already in opposition, they will exhibit simultaneous opposition in $65\frac{5}{11}$ weeks, their hour-hands being respectively at $11\frac{5}{11}$ and $9\frac{3}{11}$, whilst the hands of T, the true time-keeper, will also be in opposition at $10\frac{4}{11}$; so that all three will be in opposition together. As a further example, let $a = 3$, $b = 7$. Then $\frac{10080+3}{10} = 1008\frac{3}{10}$, and there are 10 starting-places in each primary division, or 110 in all. Thus,

with $m = 1$, $r = \frac{7}{10}$; with $m = 4$, $r = \frac{8}{10}$; with $m = 8$, $r = \frac{6}{10}$;
with $m = 2$, $r = \frac{4}{10}$; with $m = 5$, $r = \frac{5}{10}$; with $m = 9$, $r = \frac{3}{10}$;
with $m = 3$, $r = \frac{1}{10}$; with $m = 6$, $r = \frac{2}{10}$; with $m = 10$, $r = 0$.
with $m = 7$, $r = \frac{9}{10}$;

The respective times are given in these cases by $\frac{m\,(65\frac{5}{11})}{a+b} = \frac{m\,(65\frac{5}{11})}{10}$, in weeks.

If both A and B gain, or if they both lose, the time taken to effect a separation, $= k$, can be found by change of sign, $a-b$, &c., being given by $\frac{60k}{a-b}$ weeks. The process is in other respects similar. Moreover, the same law holds good for conjunction as for opposition.

[The Proposer, Mr. Bourne, and others send the following solution:—The straight-line positions of the hands succeed one another at intervals of 12/11 hours; hence the phenomenon cannot occur until one watch has gained 6/11 hours and the other has lost the same. If 1 minute is gained in 1 week, 6/11 hours will be gained in 360/11 weeks = 32 weeks 5 days 2 hours $10\frac{10}{11}$ minutes.

They consider that the above solution is like using Nasmyth's pile-driving steam hammer to crack a nut, whereon Mr. Biddle remarks that this solution is not only short but insufficient, since it takes no notice of the position of the hands at starting, which, as he has shown, makes all the difference. If the watches had started at six o'clock instead of twelve, the time occupied before simultaneous opposition of the hands next occurred would have been twice as long. This solution only shows that the time occupied could not be less than that stated. That it is the absolute or actual time in the particular instance is by no means proved in that solution.]

12123. (Prof. Fortey, M.A.)—If a series of n terms is deranged, find (1) the chance that no term stands next to a term it was next to originally; and (2) if n is infinite, prove that the chance is

$$e^{-2} = \cdot 13533528.$$

I. *Solution by* Professors Zerr, Mukhopadhyay, *and others.*

n terms can be deranged so that no term may follow it which originally followed it in $(n+1)!/ne$ ways, or the integer nearest to this value.

Also, since n terms can be arranged in $n!$ ways, the chance that no term may be followed by a term which originally followed it is

$$p_1 = (n+1)!/(nen!) = (n+1)/ne.$$

Similarly, the chance that no term may be preceded by a term that originally preceded it is $(n+1)/ne = p_2$; hence the required chance

$$p = p_1 p_2 = (n+1)^2/(n^2e^2) = (1+2/n+1/n^2)/e^2;$$

therefore $p = e^{-2}$ when n is infinite.

[The Proposer remarks that the foregoing solution is incorrect in principle, and gives a wrong result when n is a finite number. The probabilities p_1, p_2 are not those of independent events, and therefore their product does not give the probability of the happening of both.

As an illustration, suppose $n = 3$; then, writing down the permutations of *abc*, we have *abc*, *bca*, *cab*, *acb*, *cba*, *bac*, of which the last three contain no direct, and the first three no reverse, sequences; thus $p_1 = p_2 = \frac{1}{2}$, and therefore, by this solution, the chance of a row containing no sequences direct or reverse $= p_1 p_2 = \frac{1}{4}$. But this is obviously wrong, for there is *no* such row, and therefore the chance is 0.

Again, when $n = 4$, the letters *abcd* can be permuted in twenty-four ways, of which two only are free from both kinds of sequences (*i.e.*, *bdac* read either way); therefore the true chance $= \frac{1}{12}$. But the above attempt at a solution gives $p = p_1 p_2 = (\frac{11}{24})^2 = \frac{121}{576}$, or more than double the true chance.

He has not been able to express the chance as a function of n, but by an investigation too long for the volume, although quite satisfactory, he has found the value of p for values of n, from 4 to 12, and also for $n = \alpha$.

It will be seen that, as n increases, the value of p *increases* to the limit e^{-2}, whereas in the above attempt at a solution it *decreases* to the same limit.

n	No. of rows free from sequences.	p
4	2	·083333
5	14	·116667
6	90	·125000
7	646	·128175
8	5242	·130010
9	47622	·131233
10	479306	·132080
11	5296790	·132696
12	63779064	·133150
$\propto$	$\propto$	·135335 $= e^{-2}$.]

II. *Solution by* Professor Swaminatha Aiyar, M.A., *and the* Proposer.

Definition.—When two numbers r, $r+1$ come together, either in that order or in the order r, $r+1$, the combination is called a sequence.

1. Take the first r natural numbers, and $n-r$ other quantities incapable of forming sequences, either amongst themselves or with any of the natural numbers, and let ${}_nQ_r$ be the number of permutations of these n quantities in which there are *no* sequences.

2. To find ${}_nQ_{r+1}$ take any one of the $n-r$ quantities, m suppose, and in the permutations of ${}_nQ_r$ change m into $r+1$. If from the resulting permutations we take away all those in which the sequences $r,\ r+1$ or $r+1,\ r$ occur, we get ${}_nQ_{r+1}$.

3. First let us consider $r,\ r+1$. Regard this as a single number, the same as r, and subtract ${}_{n-1}Q_r$ from ${}_nQ_r$. Now, in some of the original permutations of ${}_nQ_r$, the combination $r,\ m,\ r-1$ occurs. Then m has been changed into $r+1$, and, finally, the combination $r,\ r+1$ has been reduced to the single number r.

Therefore $r,\ r+1,\ r-1$ has become $r,\ r-1$, and permutations containing this form have not been subtracted in ${}_{n-1}Q_r$, since in ${}_{n-1}Q_r$ they do not occur. Consider therefore $r,\ r+1,\ r-1$ as one number, *the same as* $r-1$, and subtract ${}_{n-2}Q_{r-1}$. Reasoning as above, it will now be found that the permutations containing $r-2,\ r,\ r+1,\ r-1$ have not been subtracted. Therefore, call this combination $r-2$, and subtract ${}_{n-3}Q_{r-2}$. For other combinations we shall have to subtract successively ${}_{n-4}Q_{r-3}$, ${}_{n-5}Q_{r-4}, \dots {}_{n-r}Q_1$. Therefore, to get rid of the sequence $r,\ r+1$, we must subtract

$$ {}_{n-1}Q_r + {}_{n-2}Q_{r-1} + {}_{n-3}Q_{r-2} + \dots + {}_{n-r}Q_1. $$

4. And precisely the same reasoning shows that we must subtract the same number of permutations to get rid of the sequence $r+1,\ r$. But the result after these subtractions is ${}_nQ_{r+1}$,

$$ \therefore \quad {}^nQ_{r+1} = {}_nR_r - 2\left\{{}_{n-1}Q + {}_{n-2}Q_{r-1} + {}_{n-3}Q_{r-2} + \dots + {}_{n-r}Q_1\right\}. $$

5. Writing $r-1$ for r, we have

$$ {}_nQ_r = {}_nQ_{r-1} - 2\left\{{}_{n-1}Q_{r-1} + {}_{n-2}Q_{r-2} + \dots + {}_{n-r+1}Q_1\right\}, $$

$$ \therefore \quad {}_{n-1}Q_{r-1} = {}_{n-1}Q_{r-2} - 2\left\{{}_{n-2}Q_{r-2} + \dots + {}_{n-r+1}Q_1\right\}, $$

$$ \therefore \quad {}_nQ_r - {}_{n-1}Q_{r-1} = {}_nQ_{r-1} - {}^{n-1}Q_{r-2} - 2{}_{n-1}Q_{r-1}, $$

or

$$ {}_nQ_r - {}_nQ_{r-1} + {}_{n-1}Q_{r-1} + {}_{n-1}Q_{r-2} = 0 \ \dots\dots\dots\dots\dots\dots \ (\alpha). $$

6. Now ${}_nQ_0 = {}_nQ_1 = n!$ and from (α) we get, in succession,

$$ {}_nQ_2 = n! - 2.(n-1)! = (1-2J)\,n! $$

$$ {}_nQ_3 = n! - 4.(n-1)! + 2(n-2)! = (1-4J+2J^2)\,n! $$

&c. = &c.

Assume, therefore, that ${}_nQ_r = A_r . n!$, where A_r is a function of J only. Then (α) becomes $(A_r - (I-J)\,A_{r-1} + JA_{r-2})\,n! = 0$.

Now let $s = A_0 + A_1x + A_2x^2 + \textit{ad inf.}$

Hence this is a recurring series with scale of relation $1-(1-J)\,x + Jx^2$,

therefore, summing, $s = \dfrac{A_0 + A_1x - (1-J)\,A_0x}{1-(1-J)\,x+Jx^2} = \dfrac{1+Jx}{1-(1-J)\,x+Jx^2}$,

since $A_0 = A_1 = 1$.

Expanding, $s = (1-\mathrm{J}x)\left\{1+(1-\mathrm{J})\,x\,(1-\mathrm{J}/[1-\mathrm{J}]\,x) + (1-\mathrm{J})^2 x^2 (1-\mathrm{J}/[1-\mathrm{J}]\,x)^2 + \&c.\right\}$;

and picking out the coefficient of x^r, we have

$$\mathrm{A}_r = (1-\mathrm{J})^r - (r-1)(1-\mathrm{J})^{r-2}\mathrm{J} + (r-2)_2(1-\mathrm{J})^{r-4}\mathrm{J}^2 - \&c.$$
$$+\left\{(1-\mathrm{J})^{r-1} - (r-2)(1-\mathrm{J})^{r-3}\mathrm{J} + (r-3)_2(1-\mathrm{J})^{r-5}\mathrm{J}^2 - \&c.\right\}\mathrm{J}\ \ldots\ (\beta),$$

where either series stops when a factor becomes 0, or an index, or factorial negative. (Note that $0! = 1$.)

Then ${}_n\mathrm{Q}_r = \mathrm{A}_r . n!$ is known.

7. Now

$${}_n\mathrm{Q}_n \mathrm{A}_n n! = \left\{(1-\mathrm{J})^n - (n-1)(1-\mathrm{J})^{n-2}\mathrm{J}^2 + (n-2)_2(1-\mathrm{J})^{n-4}\mathrm{J}^2 - \&c.\right\} n!$$
$$+\left\{(1-\mathrm{J})^{n-1} - (n-2)(1-\mathrm{J})^{n-3}\mathrm{J}^2 + (n-3)_2(1-\mathrm{J})^{n-5}\mathrm{J}^2 - \&c.\right\}(n-1)!$$

This is the number of permutations of the numbers 1 to n taken altogether in which there are *no* sequences. Therefore the chance of a permutation containing no sequences is ${}_n\mathrm{Q}_n/n!$,

and this completes the solution of the first part of the question.

8. Next, to determine the value of ${}_n\mathrm{Q}_n/n!$ when $n = \infty$.

Taking the general term in the first line of (β), we have

$$(n-s)_s(1-\mathrm{J})^{n-2s}\mathrm{J}^s . n!$$

Now, when n approaches infinity,

$$(n-s)_s = (n-s)^{(s)}/s! = n^s/s! \text{ and } \mathrm{J}^s n! = (n-s)! = n!/n^{(s)} = n!/n^s;$$

therefore $(n-s)_s\mathrm{J}^s n!$ becomes $n!/s!$, and the general term reduces to

$$1/s!\,(1-\mathrm{J})^{n-2s} n!$$

But $(1-\mathrm{J})^{n-2s}(n-2s)! = (n-2s)!\,e^{-1}_{n-2s}$ (*Choice and Chance*, 2nd ed., p. 190)

$$= (n-2s)!\,e^{-1}, \text{ when } n \text{ is very large.}$$

Therefore $(1-\mathrm{J})^{n-2s} n! = e^{-1}.n!$, and the general term is $1/s!\,e^{-1}.n!$; therefore the first line in the formula for ${}_n\mathrm{Q}_n$ is

$$(1 - 1 + 1/2! - 1/3! + \ldots)\,e^{-1}.n! = e^{-2}.n!,$$

and the second line is $\quad e^{-2}.(n-1)!$,

$$\therefore \quad {}_n\mathrm{Q}_n = [n! + (n-1)!]\,e^{-2},$$

$$\therefore \quad {}_n\mathrm{Q}_n/n! = (1+1/n)\,e^{-} = e^{-2}, \text{ since } n = \infty.$$

9. It may also be easily shown that, if r and n are both very large numbers, and the n quantities made up of the numbers 1 to r, and $n-r$ other quantities be permuted, then the chance of a permutation containing no sequences is $e^{-1-r/n}$.

[The Proposer remarks that the very original and subtle reasoning of clauses 1 to 5 is by Prof. Aiyar, and that the remainder of the solution is his own.]

2581. (M. Collins, B.A.)—In an ellipse, find (1) the locus of the middle points of all the normals, (2) the locus of the poles of the normals,

(3) the minimum normal chord, and (4) the normal chord that cuts off the least elliptic segment.

Solution by Professor LAMPE *and* W. J. GREENSTREET, M.A.

1. The normal at the point (x_1, y_1) of the ellipse $x^2/a^2+y^2/b^2=1$ has the equation $$b^2x_1(y-y_1)=a^2y_1(x-x_1),$$
and cuts the ellipse, for the second time, in the point
$$x=\frac{a^2x_1(c^4y_1^2-b^6)}{b^6x_1^2+a^6y_1^2},\quad y=\frac{b^2y_1(c^4x_1^2-a^6)}{b^6x_1^2+a^6y_1^2},\quad \text{where}\quad c^2=a^2-b^2.$$

Hence $\frac{1}{2}(x+x_1)=\xi=\dfrac{a^4c^2x_1y_1^2}{b^6x_1^2+a^6y_1^2}$; $\frac{1}{2}(y+y_1)=\eta=-\dfrac{b^4c^2x_1^2y_1}{b^6x_1^2+a^6y_1^2}$.

Eliminating x_1 and y_1 from these expressions for the coordinates ξ, η of the middle point of the normal, we find at first $\xi/\eta=-a^4y_1/b^4x_1$;

then $$x_1^2=\frac{a^8\eta^2}{a^6\eta^2+b^6\xi^2},\quad y_1^2=\frac{b^8\xi^2}{a^6\eta^2+b^6\xi^2};$$

finally substituting x_1^2 and y_1^2 in the equation of the ellipse, we get
$$(1)\quad (a^6\eta^2+b^6\xi^2)(a^2\eta^2+b^2\xi^2)^2=a^4b^4c^4\xi^2\eta^2,$$
or in polar coordinates r, θ,
$$r^2(a^6\sin^2\theta+b^6\cos^2\theta)(a^2\sin^2\theta+b^2\cos^2\theta)^2=a^4b^4c^4\sin^2\theta\cos^2\theta.$$
This shows the locus to have four loops, or four leaves uniting in the centre of the ellipse.

Working out $(x_1-x)^2+(y_1-y)^2=n^2=$ square of the length of the normal chord, we find $n^2=\dfrac{4b^2(a^4-c^2x_1^2)^3}{[a^6-x_1^2(a^4-b^4)]^2}$.

2. Treating this expression as a function of x_1, the minimum of n is determined at (3) $x_1^2=\dfrac{a^4(a^2-2b^2)}{a^4-b^4}$, $y_1^2=\dfrac{b^4(2a^2-b^2)}{a^4-b^4}$, $n=\dfrac{3a^2b^2\sqrt{3}}{(a^2+b^2)^{\frac{3}{2}}}$;

i.e., only possible if $a>b\sqrt{2}$. In this case x_1 and y_1 are the coordinates of that point of the ellipse whose osculating circle has its centre on the ellipse.

The equation of the locus of the poles of all normals is easily found to be
$$(2)\quad \frac{a^6}{x^2}+\frac{b^6}{y^2}=c^4;$$
this is the equation of a curve called "kreuz curve."

3. All chords of an ellipse which cut off a constant area envelop a similar, similarly placed, and concentric ellipse (see SALMON, *Conic Sections*, § 396). The greater this interior ellipse is taken, the smaller is the area cut off, Our problem will be solved, if we come to determine the greatest similar and concentric ellipse touched by a normal. But all normals envelop the evolute of the ellipse; henceforth we must investigate the ellipse $x^2/a^2+y^2/b^2=k^2$ which touches the evolute of the ellipse given. If x

and y are the coordinates of a point of the original ellipse, the coordinates of its centre of curvature are $\alpha = \frac{c^2x^3}{a^4}$, $\beta = -\frac{c^2y^3}{b^4}$.

Putting these values into the equation $x^2/a^2+y^2/b^2 = k^2$, we obtain

$$\frac{c^4x^6}{a^{10}}+\frac{c^4y^6}{b^{10}} = k^2, \quad \text{or} \quad \frac{c^4x^6}{a^{10}}+\frac{c^4(a^2-x^2)^3}{a^6b^4} = k^2.$$

This cubic equation for x^2 gives us the intersection of the evolute with the interior ellipse. If both curves touch, this equation has two equal roots, or its first derivative is zero, $\frac{6c^4x^5}{a^{10}} - \frac{6c^4(a^2-x^2)^2x}{a^6b^4} = 0$.

Solving for x, we find (4) $x^2 = \frac{a^4}{a^2+b^2}$, $y^2 = \frac{b^4}{a^2+b^2}$, $k^2 = \left(\frac{a^2-b^2}{a^2+b^2}\right)^2$.

Therefore the axes of the concentric ellipse are

$$a\frac{a^2-b^2}{a^2+b^2},\quad b\frac{a^2-b^2}{a^2+b^2};\quad \alpha = \frac{a^2c^2}{(a^2+b^2)^{\frac{3}{2}}},\quad \beta = -\frac{b^2c^2}{(a^2+b^2)^{\frac{3}{2}}}.$$

The area of the least segment cut off by the normal proves to be

$$\tfrac{1}{2}ab\left(\arctan\frac{2ab}{a^2-b^2} - \frac{2a^2b^2(a^2-b^2)}{(a^2+b^2)^2}\right).$$

4. In Professor Greenhill's *Diff. and Int. Calculus*, 2nd ed., p. 204, No. (8), we read: "Prove that the normal chord which divides the area of an ellipse most unequally is equally inclined to the axes." This proposition follows immediately from the expression giving the tangent of the angle formed by the normal chord and the axis of x, viz., $a^2\alpha/b^2\beta$, $= -1$. The same problem was enunciated by Prof. Lemoine in the *Nouvelle Correspondance Mathématique* as Question 384, and solved by Catalan as No. 464 (in 1880). I got this information from Mr. Laisant's *Recueil de problèmes de Mathématiques, Géométrie analytique à deux dimensions* (Paris, 1893), p. 65, No. 250. This work contains moreover the notice (p. 186, No. 729) that in 1868 Paillotte treated our Question 2581 in the *Nouvelles Annales* of the year 1868 as No. 519. Unacquainted with these facts, I sent the solution which I had prepared for my lessons on the Calculus, and till now I took no opportunity of comparing the passages quoted above.

5. The same questions for the parabola $y^2 = 2px$.

(1) Equation of the locus of all middle points, $x = p+p^3/2y^2+y^2/p$.

(2) Shortest normal, $x_1 = p$, $y_1 = p\sqrt{2}$, $n = p\sqrt{3}$.

(3) Locus of poles of all normals, $2y^2(p+x)+p^3 = 0$.

(4) Normal chord that cuts off the least parabolic segment, $x_1 = \frac{1}{2}p$, $y_1 = p$, *i.e.*, the normal at the end of the latus rectum. Area of the least segment, $A = \frac{16}{3}p^2$.

11800. (Editor.)—Solve the equations

$$x^{-1}+(y-z)^{-1} = (b+c)^{-1},\quad y^{-1}+(z-x)^{-1} = b^{-1},\quad z^{-1}+(x-y)^{-1} = c^{-1}.$$

Solution by H. J. WOODALL, A.R.C.S.

We have $(x+y-z)(b+c) = x(y-z)$, $(y+z-x)b = y(z-x)$,

$$(z+x-y)c = z(x-y) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1, 2, 3).$$

By addition, $2by+2cx = 0$; $\therefore x : y = -b : c$, say $x = -kb$, $y = kc$;

from (1), (2), $z(x-b-c) = xy-(x+y)(b+c)$, $z(b-y) = b(x-y)-xy$;

whence $(b-y)xy-(b-y)(x+y)(b+c) = (x-b-c)b(x-y)-xy(x-b-c)$;

substituting for x, y,

$$-(b-kc)k^2bc-(b-kc)(-kb+kc)(b+c)$$
$$= +(kb+b+c)bk(b+c)-k^2bc(kb+b+c),$$

or $$k\left\{k^2bc(b+c)+k(c^3+bc^2-2b^2c-b^3)-2bc(c+b)\right\} = 0.$$

But from the given equations we see that the solution corresponding to $k=0$ is extraneous; hence there are two sets of values which can be found from $k^2bc(b+c)+k(c^3+bc^2-2b^2c-b^3)-2bc(c+b) = 0$; and therefrom x, y, z by substituting in any one of the given equations.

12277 & 12315. (D. BIDDLE.)—Solve the following equations :—

$$x^3-12x^2+24x-16 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$x^6-60x^5+450x^4-1800x^3+4050x^2-4860x+2430 = 0 \quad \ldots\ldots \quad (2),$$

$$x^3-150x-247 = 0, \quad x^3-19x-1950 = 0 \ldots\ldots\ldots\ldots \quad (3, 4).$$

Solution by J. J. BARNIVILLE, B.A.; T. SAVAGE; *and others.*

Substituting $2v$ for x in (1), and $3v$ for x in (2), we have

(1) $(v-1)^3 = \frac{1}{2}v^3$, whence $x = 2/(1-2^{\frac{1}{3}})$;

(2) $(v-1)^6 = \frac{7}{10}v^6$, whence $x = 3/(1\mp\sqrt[6]{\frac{7}{10}})$;

(3) $x^3-150x-247 = (x-13)(x^2+13x+19)$;

(4) $x^3-19x-1950 = (x-13)(x^2+13x+150)$.

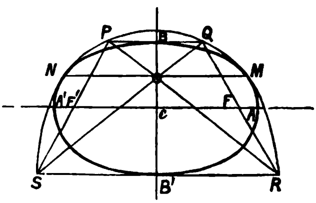
In general, $x^3+(b-a^2)x-ab = (x-a)(x^2+ax+b)$.

From this it will be seen that, as the PROPOSER remarks, (1) and (2) are of a similar type; and that, from the general formula added, any number of similar equations may, as he wishes, be derived.

[The PROPOSER observes that the solutions are correct, but due rather to trial than to a method both "vigorous and rigorous."]

11758. (Professor TISSOT.)—Un cercle enveloppe une ellipse et la touche en deux points réels; un trapèze inscrit dans ce cercle a ses côtés parallèles sur les tangentes à l'ellipse aux extrémités du petit axe. Démontrer que chacun des côtés non parallèles du trapèze passe par l'un des foyers, et que chaque diagonale est parallèle à l'une des droites qui joignent les foyers aux extrémités du petit axe. Lorsque les points de contact de l'ellipse avec le cercle sont imaginaires, ce sont les diagonales du trapèze qui passent par les foyers, et ce sont les côtés non par llèles qui ont les directions indiquées.

Solution by W. J. GREENSTREET, M.A.; Professor CHAKRIVARTI; *and others.*



Let any chord MN be $Ax+By-1$; then the general equation of conics touching (E),

$$x^2/a^2+y^2/b^2-1=0,$$

at its intersection with AB is

$$\lambda\,(x^2/a^2+y^2/b^2-1)+(Ax+By-1)^2=\ldots0\ldots(1).$$

This is a circle (1) if $AB=0$, *i.e.*, MN is parallel to one or other of the axes. In the figure MN is parallel to the major axis, and the circle is without the ellipse. If MN be parallel to the minor axis, the circle is within the ellipse.

Let MN be $\qquad By-1=0.$

(2) Then (1) is a circle if $\lambda c^2=1$, and (1) becomes

$$(x^2+y^2)\,b^2+(2c^2/B)\,y-(a^2b^2+c^2/B^2)=0\ldots\ldots\ldots\ldots\ldots(C).$$

The tangents at the extremities of the minor axes are $y^2-b^2=0$. Any conic through the intersections of (C) and $(y^2-b^2)=0$ is

$$b^2\,(x^2-y^2)+(2c^2/B)\,y-(a^2b^2+c^2/B^2)+\mu\,(y^2-b^2)=0\ldots\ldots\ldots(\Delta).$$

This represents two straight lines if

$$\mu^2b^2+\mu\,(a^2b^2+c^2/B^2+b^4)+a^2\,(b^4+c^2/B^2)=0,$$

or
$$\mu_1=(-c^2/B^2+b^4)/b^2;\quad \mu_2=-a^2.$$

Inserting the values of μ_1 and μ_2 in (Δ), we get

$$(y/B-b^2)^2-b^4x^2/c^2=0\ \text{ and }\ b^2x^2-c^2\,(y-1/B)^2=0\ldots\ldots\ldots(3,4).$$

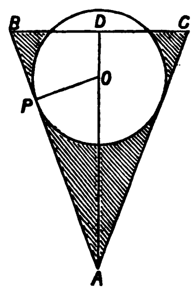
The lines (3) pass through $\pm c$, 0. These are the non-parallel sides. The lines (4) make angles $\tan^{-1}(\pm b/c)$ with the axes, and are therefore parallel to FB, FB′. They intersect at the middle point O of MN on CB.

(3) The trapezium is real as long as $1/B<b$; when $Bb>1$, the lines (3), (4) are real. But P, Q, R, S are imaginary.

863. (EDITOR.)—If a conical glass, whose altitude is a and the generating angle θ, be filled with water, find the radius of the sphere

which, being put into it, shall cause the greatest quantity of water to overflow.

Solution by Professors Zerr, Mukhopadhyay, *and others.*



Let OD $= z$, OP $= x =$ radius required. From the similar triangles ABD, AOP, we have $z = (a\sin\theta - x)/\sin\theta$,

$\frac{1}{3}\left[2\pi x^2(x-z) - \pi(x^2-z^2)z\right]$ = volume of segment of sphere outside the cone;

$\therefore\ \frac{1}{3}\pi\left[4x^3 - 2x^2(x-z) + (x^2-z^2)z\right]$ = volume of segment within cone = maximum;

substituting value of z and differentiating, we get $x = a\sin\theta/(1-2\sin^2\theta+\sin\theta)$.

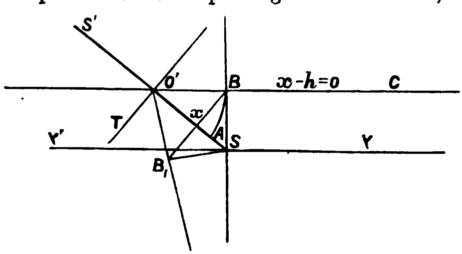
If $\theta = 30°$, $x = \frac{1}{3}a$.

10837. (Professor Morel.)—On donne deux axes rectangulaires Ox et Oy, et une droite fixe BC, parallèle à OY, et dont l'équation est $x - h = 0$. On considère toutes les hyperboles ayant un foyer à l'origine, et pour asymptote la droite BC. Trouver (1) le lieu géométrique du second foyer; (2) le lieu géométrique du point de rencontre de l'axe, avec la directrice correspondant au premier foyer; (3) l'enveloppe de la seconde asymptote; (4) l'enveloppe de l'axe non transverse; (5) le lieu des sommets réels; (6) le lieu du pied de la seconde directrice sur l'axe transverse.

Solution by H. J. Woodall, A.R.C.S.; Professor Chakrivarti; *and others.*

(1) The second focus is as far above BC as S is below; therefore the locus is right line $x - 2h = 0$.

(2) The perpendicular from focus on to the asymptote meets that asymptote at a point on the corresponding directrix. Also, if x be inter-



section of directrix and transverse axis SxB = right angle, therefore locus of x is a circle on SB diameter, *i.e.*, $x^2 - hx + y^2 = 0$.

(3) Perpendiculars from S on the asymptotes are equal; therefore envelope of second asymptote is circle $x^2+y^2 = h^2$.

(4) Envelope of non-transverse axis is parabola focus S, vertex at B, *i.e.*, envelope is $y^2+4h(x-h) = 0$.

(5) Put angle $Y'SO' = \theta$; then locus of A, A′ is (since $O'A = O'B$)

$$r \mp O'A = h \operatorname{cosec}\theta;$$

therefore $\quad r = h(\operatorname{cosec}\theta \pm \cot\theta) = h\cot\tfrac{1}{2}\theta$ or $h\tan\tfrac{1}{2}\theta$.

(6) So locus of foot of second directrix is $r = h(\operatorname{cosec}\theta + \cot^2\theta)$.

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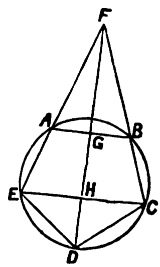
601. (S. Watson.)—ABCDE is a regular pentagon, and its sides EA, CB are produced to meet in F; FD, CE are joined; then, if the triangles CDE, AFB revolve about the common axis DF, the convex surfaces and volumes generated by these triangles will be equal.

Solution by Professors G. B. M. Zerr, Bhattacharya, *and others.*



Let $AB = a =$ side of polygon, then

$$AF = \tfrac{1}{2}(5^{\frac12}+1)\,a, \quad FG = \tfrac{1}{2}(5+2\,5^{\frac12})^{\frac12}\,a,$$

$$EH = \tfrac{1}{4}(5^{\frac12}+1)\,a, \quad DH = \tfrac{1}{4}(10-2\,5\ )^{\frac12}\,a.$$

Convex surfaces of AFB, EDC are

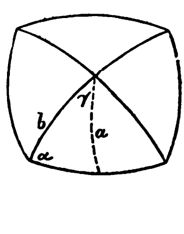
$$\tfrac{1}{4}\pi a^2(5^{\frac12}+1), \quad \tfrac{1}{4}\pi a^2(5^{\frac12}+1);$$

volumes of AFB, EDC are

$$\tfrac{1}{24}\pi a^3(5+2\,5^{\frac12})^{\frac12}, \quad \tfrac{1}{24}\pi a^3(5+\quad ^{\frac12})^{\frac12}$$

1215. (Editor.)—Find the square which, when placed upon a sphere, and four planes drawn through its sides and the centre of the sphere, shall cut off a given surface.

Solution by Professor Lampe.



Let the square touch the sphere in the crossing-points of its diagonals; then the four planes passing through its sides intersect the sphere in an equiangled and equisided spherical quadrilateral.

If $2s$ is the side of the square required, R the radius of the sphere, b half the diagonal of the spherical quadrilateral, a the spherical distance of its centre from the side, α the angle included by b

and the side, γ the angle included by b and a; then

$$\tan a = s/\mathrm{R}, \quad \tan b = (s\sqrt{2})/\mathrm{R}, \quad \cot a \sin b = \cot \alpha \sin \gamma + \cos b \cos \gamma\,;$$

or, γ being $= \frac{1}{4}\pi$, $\quad \cot \alpha = a \sin b \sqrt{2} - \cos b$.

But $\cot a = \mathrm{R}/s, \quad \cos b = \mathrm{R}/(\mathrm{R}^2 + 2s^2)^{\frac{1}{2}}, \quad \sin b = (s\sqrt{2})/(\mathrm{R}^2 + 2s^2)^{\frac{1}{2}}$;

whence $$\cot \alpha = \mathrm{R}/(\mathrm{R}^2 + 2s^2)^{\frac{1}{2}}.$$

Solving this expression for s, we find

$$s = (\mathrm{R}/\cos \alpha)\,(-\tfrac{1}{2}\cos 2\alpha)^{\frac{1}{2}}.$$

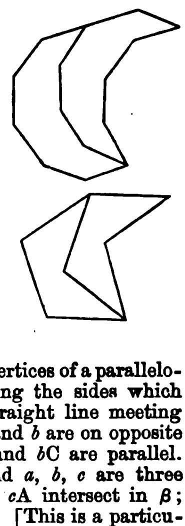
Now let the area of the spherical quadrilateral be $= \mathrm{A}$; then

$$\mathrm{A} = \mathrm{R}^2(8\alpha - 2\pi), \quad \text{or} \quad \alpha = \tfrac{1}{8}\left\{(\mathrm{A}/\mathrm{R}^2) + 2\pi\right\},$$

to be substituted in the expression of s.

12285. (F. R. J. Hervey.)—A way of dividing a regular polygon of $2n$ sides into as many parts exactly alike, not triangles or kites, each having an axis of symmetry, is given at page 59 of Vol. LIII. Find another way when n is odd.

Solution by the Proposer.



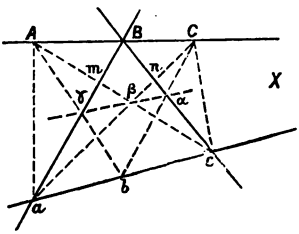
Fold a polygon of m sides along one of the diagonals which divide it not equally but least unequally.

If m be even, m of the figures so obtained will make a polygon of m sides. This is the way of Question 10269, Vol. LIII.

But if m be odd, $2m$ such figures make a polygon of $2m$ sides.

12322. (W. J. Dobbs, B.A.)—A, a are opposite vertices of a parallelogram; through A is drawn any straight line meeting the sides which intersect at a in B and C; through a is drawn any straight line meeting the sides which intersect at A in b and c; so that B and b are on opposite sides, C and c on opposite sides; prove that (1) Bc and bC are parallel. Hence (2) if A, B, C are three collinear points, and a, b, c are three collinear points, Bc and bC intersect in α; Ca and cA intersect in β; Ab and aB intersect in γ: then α, β, γ are collinear. [This is a particular case of Pascal's theorem. In $\begin{vmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} \\ a & b & c \\ \alpha & \beta & \gamma \end{vmatrix}$ each point is the intersection of the straight lines represented by the corresponding minor. If any two rows are collinear, so is the remaining one.]

Solution by Professors DROZ-FARNY, BHATTACHARYA, *and others.*

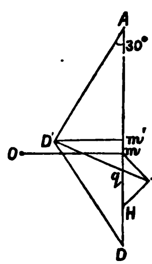


2. Représentons par x le point de coupe des droites ABC et abc. On a faisceau A $(abcx)$ homographique au faisceau C $(abcx)$. La transversale Ba rencontre les rayons du premier faisceau aux points a, γ, m, B et la transversale Bc coupe les rayons du second suivant les points n, a, c, B ; il en résulte que les ponctuelles a, γ, m, B et n, a, c, B sont homographiques et comme elles ont en B, 2 points correspondants en commun elles sont perspectives. Les droites na, mc, et γa se croisent au même point β.

1. Dans le parallélogramme Ac et Ca ainsi que Ab et Ba sont parallèles donc β et γ sont à l'infini. La droite $a\beta\gamma$ est donc à l'infini d'où le parallélisme de Bc et bC.

12227. (Professor MORLEY.) — The centres (1) of an equilateral triangle described on one side of a triangle T, inwards; (2) of an equilateral triangle described on another side, outwards; of one of the equilateral triangles of which the orthocentre and circumcentre are vertices—these three are collinear.

Solution by Professors RAMASWAMI AIYAR, IGNACIO BEYENS, *and others.*



Let H, O be the orthocentre and circumcentre of the triangle ABC. If equilateral triangles (directly similar) be described on AB, AC, HO, their centres shall be collinear.

Draw AD perpendicular on BC; let D′ be the centre of the equilateral triangle described on AD; draw through D′ a line perpendicular to AD′ cutting AD in q; let Om, D′m' be perpendiculars on AD; and Hn perpendicular on the line D′q.

Since AD is the perpendicular on BC, it follows that the centres of the equilateral triangles described on AB, AC lie on the line D′q.

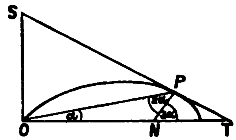
Again, $qm' = \frac{1}{2}q\mathrm{D}' = \frac{1}{2}q\mathrm{D}$; and

$$m'm = \mathrm{A}m - \mathrm{A}m' = \tfrac{1}{2}(\mathrm{AD}+\mathrm{HD}) - \tfrac{1}{2}\mathrm{AD} = \tfrac{1}{2}\mathrm{HD}.$$

Hence $qm = \frac{1}{2}q\mathrm{H}$; and therefore n is the centre of an equilateral triangle described on Hm. And, as mO is perpendicular to Hm, it follows that the centre of the equilateral triangle inscribed on HO lies in the line through n perpendicular to Hn; that is, in the line D′q, which proves the theorem.

12299. (Professor LAMPE.)—If P be a point of the Bernoullian lemniscate $r^2 = a^2 \cos 2\phi$, ST the tangent at P intersecting the axis of x in T, the axis of y in S, PN the normal at P cutting the axis of x in N, prove that (1) the area of triangle OST is minimum for $\phi = 22\frac{1}{2}^\circ$; (2) the normal PN is a maximum for $\phi = 30^\circ$; (3) the tangent ST is a minimum for $\cos 4\phi = \frac{1}{4}\left[-3 + (17)^{\frac{1}{2}}\right]$; (4) the area of the triangle OPN is a maximum for $\cos 2\phi = \cdot 5651977$; (5) time of descent on ST is a minimum for $\cos 2\phi = \cdot 892683$; on PN for $\phi = 30^\circ$, and a maximum for $\cos 2\phi = \frac{1}{4}$.

Solution by C. MORGAN, M.A., R.N.



(1) $\quad OT = a \,.\, (\cos 2\alpha)^{\frac{3}{2}}/\cos 3\alpha,$

$OS = a \,.\, (\cos 2\alpha)^{\frac{3}{2}}/\sin 3\alpha;$

$\therefore \quad \Delta OST \propto \cos^3 2\alpha/\sin 6\alpha.$

For a minimum,

$\cos(6\alpha - 2\alpha) = 0, \quad \alpha = 22\frac{1}{2}^\circ.$

(2) $\quad PN = a \sin\alpha\,(\cos 2\alpha)^{\frac{1}{2}}/\sin 3\alpha.$

For a maximum, $\quad -4\cos 2\alpha + 3 - 4\sin^2\alpha = 0, \quad$ whence $\quad \alpha = 30^\circ.$

(3) $\quad ST = a \cos^{\frac{3}{2}} 2\alpha/\sin 6\alpha.$

For a minimum, $\cos 8\alpha + 3\cos 4\alpha = 0$, whence $\cos 4\alpha = \frac{1}{4}(-3 + 17^{\frac{1}{2}})$.

(4) $\quad \Delta OPN \propto (\sin 4\alpha \,.\, \sin\alpha)/\sin 3\alpha, \quad \propto (\cos 3\alpha - \cos 5\alpha)/\sin 3\alpha.$

For a maximum, $\quad 2\cos^3 2\alpha + 2\cos^2 2\alpha = 1; \quad \therefore \cos 2\alpha = \cdot 5651977.$

(5) $\quad (\text{Time of descent on PN})^2 \propto \sin\alpha \cos 2\alpha^{\frac{1}{2}}/\sin^2 3\alpha,$

whence $\quad 0 = \sin 3\alpha \cos 3\alpha - 6\sin\alpha \,.\, \cos 2\alpha \,.\, \cos 3\alpha;$

$\therefore \quad \sin\alpha = 0, \qquad \cos 3\alpha = 0, \qquad 0 = 3 - 4\sin^2\alpha - 6\cos 2\alpha;$

$\alpha = 0, \qquad \alpha = 30^\circ, \qquad \therefore \cos 2\alpha = \frac{1}{4}$

gives a maximum, gives a minimum, gives a maximum.

$(\text{Time of descent on ST})^2 \propto \cos^{\frac{3}{2}} 2\alpha/\sin 6\alpha \cos 3\alpha,$

whence $\quad \cos 2\alpha - 2\cos 4\alpha - \cos 8\alpha = 0,$

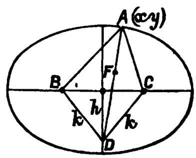
or $\quad 8\cos^4 2\alpha - 4\cos^2 2\alpha - \cos 2\alpha - 1 = 0,$

a biquadratic equation for $\cos 2\alpha$ with the root

$\cos 2\alpha = \cdot 892683, \quad \alpha = 13^\circ 23' 37,8''.$

12209. (R. CHARTRES.)—If B, C are foci of an ellipse, and A a point on the curve, and F Fermat's point, so that Σ(FA) is a minimum, find the maximum value of this minimum as A moves on the curve, and the condition that a maximum is possible. Also, if O is the centroid of the triangle ABC, find the minimum value of the minimum $\Sigma(\text{OA})^2$.

Solution by the PROPOSER.



Describe an equilateral triangle DBC on BC; then Fermat's point is the intersection of AD and the circumcircle of BDC, and Σ (FA) = AD. This I gave in *Nature*, 1888. See Vol. LVI.

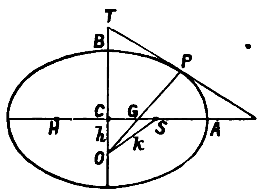
We have, now, only to find the maximum value of AD, or of $x^2 + (y+h)^2$, with the condition $x^2/a^2 + y^2/b^2 = 1$, either by the calculus or by eliminating x^2.

This gives $\qquad AD = k/e = 2ae/e = 2a.$

If G be the centroid, then $\Sigma\,(GA^2) = \frac{1}{3}\Sigma\,(AB^2)$ has its minimum value when $AB = AC = a$, since $AB + AC = 2a$;

$$\therefore\ \text{minimum} = \tfrac{1}{3}(2a^2 + 4a^2e^2) = 2a^2/3\,(1 + 2e^2).$$

The following addition is, I think, new:—



Let $OC = h$, $OS = k$; then OP, the maximum straight line from any point O in the axis minor, is normal to the tangent at P.

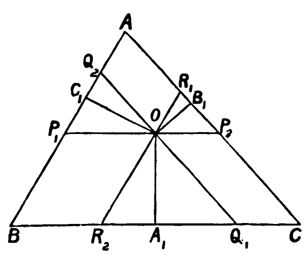
$$OP = k/e, \quad OG = ke, \quad OT = k^2/h,$$

semi-conjugate to $CP = bk/ae$.

Also OT will be constant for a system of confocal conics, and we have at once their principal properties.

1276. (EDITOR.)—Find (1) the locus of a point such that, if parallels be drawn through it to the three sides of a triangle, the sum of the rectangles under the three pairs of intercepts on each line respectively, between the point and the two sides which it meets, shall be equal to a given rectangle; and (2) what is the position of the point *within* the triangle, when the sum of the three rectangles is a maximum.

Solution by Professor LAMPE.



Let P_1P_2, Q_1Q_2, R_1R_2 be respectively parallel to BC, CA, AB, and $OA_1 = x$, $OB_1 = y$, $OC_1 = z$ perpendicular to BC, CA, AB, $= a, b, c$. Then, calling the radius of the circumscribed circle r, we have

$$OP_1 \,.\, \sin B = z,$$

$$OP_1 = z/\sin B = 2rz/b; \quad OP_2 = 2ry/c;$$

or $\qquad OP_1 \,.\, OP_2 = 4r^2yz/bc;$

$$OQ_1 \,.\, OQ_2 = 4r^2zx/ca;$$

$$OR_1 \,.\, OR_2 = 4r^2xy/ab;$$

$\therefore \quad OP_1 . OP_2 + OQ_1 . OQ_2 + OR_1 . OR_2 = 4r^2/abc\,(ayz + bzx + cxy).$

Designating by Δ the area $abc/4r$ of the triangle ABC, and by q^2 a given constant, we get $ayz + bzx + cxy = q^2\Delta/r$. This is the equation of a circle concentric with the circumscribing circle (SALMON's *Conics*, § 125).

To investigate the maximum or minimum of

$$r/\Delta\,(ayz + bzx + cxy) \qquad \text{.............................(1)},$$

with

$$ax + by + cz - 2\Delta = 0 \qquad \text{.............................(2)},$$

form $\quad f(x, y, z) = r/\Delta\,(ayz + bzx + cxy) - \lambda\,(ax + by + cz - 2\Delta).$

The conditions $\partial f/\partial x = 0$, $\partial f/\partial y = 0$, $\partial f/\partial z = 0$, prove to be

$$r/\Delta\,(\,.\ cy + bz) = \lambda a, \quad r/\Delta\,(cx\ .\ + az) = \lambda b, \quad r/\Delta\,(bx + ay\ .\,) = \lambda c.$$

Wherefrom $\quad x = \Delta/r\,\lambda\cos A, \; y = \Delta/r\,\lambda\cos B, \; z = \Delta/r\,\lambda\cos C.$

Putting these values into (2), we obtain

$$\Delta\lambda/r\,(a\cos A + b\cos B + c\cos C) = 2\Delta;$$

but $a\cos A + b\cos B + c\cos C$ being $= 8\Delta^2/abc$, we have $\lambda = r^2/\Delta$, and $x = r\cos A$, $y = r\cos B$, $z = r\cos C$, or in the case of the minimum of q^2 we must choose O as the centre of the circumscribing circle, and then q^2 is found to be $= r^2$.

12278. (W. W. TAYLOR, M.A.)—In an examination, boys are asked to assign genders to each of ten words. Three genders being equally likely, find the relative probabilities that a boy, who knows none, will get 1, 2, 3, 4, to 10 right, by dint of answering them all.

Solution by Rev. C. M. SANDERSON, M.A.

His chance of getting one right and all the rest wrong will be $10 \times \frac{1}{3} \times (\frac{2}{3})^9$, and similarly his chance of getting two right and all the rest wrong will be $\dfrac{10\,.\,9}{1\,.\,2} + \left(\dfrac{1}{3}\right)^2 \times \left(\dfrac{2}{3}\right)^8$. Thus his several chances (including the chance of getting all wrong) will be represented by the terms of the expansion of the binomial $(\frac{2}{3} + \frac{1}{3})^{10}$, and the required relative chances will be 5120, 11520, 15360, 13440, 8064, 3360, 960, 180, 20, 1.

1238. (EDITOR.)—A conic passes through the angular points and the *centroid* of a given triangle; find the area of the locus of its centre.

Solution by H. W. CURJEL, M.A.; R. TUCKER, M.A.; *and others.*

The equation to any conic through A, B, C and the centroid is $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$, where $l\alpha + m\beta + n\gamma = 0$; and the centre is given by

$$\frac{m\gamma + n\beta}{a} = \frac{n\alpha + l\gamma}{b} = \frac{l\beta + m\alpha}{c} = \frac{2l\beta\gamma}{b\beta + c\gamma - a\alpha} = \&c.$$

Hence the locus of the centre is

$$\Sigma\left\{a\alpha(c\gamma+b\beta-a\alpha)\right\}=0,\quad i.e.,\quad \Sigma a^2\alpha^2=2\Sigma bc\beta\gamma,$$

which is a conic touching the sides of the ΔABC at their middle points. Hence its area $=\frac{1}{9}\pi\sqrt{3}$ (area of ΔABC); since the ΔABC and this conic may be projected orthogonally into an equilateral triangle and its in-circle.

1019. (J. DIXON.)—A cubic inch of glass is blown into the form of a globe that will hold one pint of wine. Show that the thickness of the glass is ·02159 inch.

Solution by H. W. CURJEL, B.A.; K. S. PUTNAM; *and others.*

Let r = the internal radius, and t = the thickness, in inches. Then

$$\frac{4\pi}{3}r^3=34{\cdot}65925\quad\text{and}\quad\frac{4\pi}{3}(r+t)^3=35{\cdot}65925;$$

$$\therefore\quad t=\left(\frac{3}{4\pi}\right)^{\frac{1}{3}}\left\{(35{\cdot}65925)^{\frac{1}{3}}-(34{\cdot}65925)^{\frac{1}{3}}\right\}^{\frac{1}{3}}=2{\cdot}04419-2{\cdot}02260\text{ nearly}$$
$$={\cdot}02159\text{ inch nearly.}$$

4091. (The late M. COLLINS, LL.D.)—Prove that

$$\frac{16N+20N^2D+5D^2}{16N^4+12N^2D+D^2}N$$

is nearer to $(N^2+D)^{\frac{1}{2}}$ than any rational fraction in its lowest terms n/d when $d<16N^4+12N^2D+D^2$ and $D=\pm1$; and show more generally that this theorem if true, whatever be the value of D, if

$$dD<16N^4+12N^2D+D^2.$$

Solution by H. J. WOODALL, A.R.C.S.

Expanding $(N^2+D)^{\frac{1}{2}}$ as a continued fraction, we get

$$(N^2+D)^{\frac{1}{2}}=N+\frac{1}{2N/D}+\frac{1}{2N}+\frac{1}{2N/D}+\&c.,$$

to whose value we get the following convergents, viz.,

$$1/0,\quad N/1,\quad \left(\frac{2N^2}{D}+1\right)\Big/\frac{2N}{D},\quad \left(\frac{4N^3}{D}+3N\right)\Big/\left(\frac{4N^2}{D}+1\right),$$
$$\left(\frac{8N^4}{D^2}+\frac{8N^2}{D}+1\right)\Big/\left(\frac{8N^3}{D^2}+\frac{4N}{D}\right),$$

and
$$\left(\frac{16N^5}{D^2}+\frac{20N^3}{D}+5N\right)\Big/\left(\frac{16N^4}{D^2}+\frac{8N^2}{D}+\frac{4N^2}{D}+1\right);$$

this last becomes $N(16N^4+20N^2D+5D^2)/(16N^4+12N^2D+D^2)$. This is, by the theory of continued fractions, nearer the value of $(N^2+D)^{\frac{1}{2}}$ than any fraction n/d where $d < 16N^4+12N^2D+D^2$ and $D = \pm 1$.

It would seem that the second part of the Question is disposed of by the following quotation from CHRYSTAL's *Algebra*:—"In a continued fraction of the first class (*i.e.*, such as that considered above), the odd convergents form an increasing series, and the even convergents a decreasing series; and every odd convergent is less than, and every even convergent is greater than, the following convergent."

10905. (A. MARTIN, M.A., LL.D.)—If four pennies be piled up at random on a horizontal plane, find the probability that the pile will stand.

Solution by H. W. CURJEL, B.A.

Since the rotation of the higher part of the pile, consisting of any number of pennies about a vertical axis through its centre of gravity, does not affect the equilibrium of the pile, we need consider only the positions of the centres of gravity of the parts of the pile, independent of rotations about vertical axes through their centres of gravity.

Let all the coins except the top one be held so as not to fall.

Then the range of the centre of the top coin $= \pi(2r)^2$ where $r =$ the radius of a penny and range for equilibrium $= \pi r^2$; therefore chance that the top coin will stand when the rest are held $= \frac{1}{4}$.

Similarly the total range of the centre of gravity of the two top coins $= 4\pi r^2$, if the lower coins are held and the line joining the centres of the two top coins has a constant direction, and the range for equilibrium $= \pi r^2$; therefore chance that the two top coins will stand when the rest are held $= (\frac{1}{4})^2$.

In the same manner, the chance that the n top coins will stand if the rest are held $= (\frac{1}{4})^n$; *i.e.*, chance for equilibrium of a pile of $n+1$ coins $= (\frac{1}{4})^n$; therefore chance for 4 coins $= \frac{1}{64}$.

12245. (J. L. MACKENZIE, B.A., B.Sc.)—If x^2+Ax+B and x^2+Ax-B (where A and B are positive integers) are both resolvable into simple factors, show that there are only two sets of values for A and B, namely, (1) $A = 5t$, $B = 6t$; (2) $A = 13t$, $B = 30t$.

Solution by C. W. BOURNE; R. F. DAVIS, M.A.; *and others.*

If $x^2+Ax+B = (x+\alpha)(x+\beta)$, then x^2+Ax-B will be resolvable into simple factors if $\alpha^2+\beta^2+6\alpha\beta$ is a perfect square. This requires α, β to be of the forms $6n^2-2mn$ and m^2-n^2, respectively.

Taking $m = 6$, $n = 5$, we have $(x+90)(x+11) = x^2+101x+990$, $(x-9)(x+110) = x^2+101x-990$, so that the Question appears incorrect.

11886. (ELIZABETH BLACKWOOD.)—An indefinitely large plane area is ruled with parallel equidistant straight lines; a is the distance between two consecutive lines. A second set of parallel equidistant straight lines crosses the former set at right angles; b is the distance between two consecutive lines of the second set. A regular polygon is then thrown down on the area; the polygon has $4m$ sides, and the diameter of the circle circumscribing the polygon is less than a and also less than b. Determine the chance that the polygon will fall across a line.

Solution by Professor ZERR.

Let $2r =$ the diameter of the circumscribing circle, and let θ be the angle it makes with one of the first set of lines; then, if the centre falls upon the area $ab-(a-2r\cos\theta)(b-2r\cos\theta)$, the polygon will fall across a line;

$$\therefore \quad \text{chance} = p = \int_0^{\pi/4m} \left\{ab-(a-2r\cos\theta)(b-2r\cos\theta)\right\} d\theta \Big/ \int_0^{\pi/4m} ab\, d\theta$$

$$= \frac{4m}{\pi ab}\int_0^{\pi/4m}\left\{2r(a+b)-4r^2\cos\theta\right\}\cos\theta\, d\theta$$

$$= \left\{(a+b)\,l-r\left(2\pi r+l\cos\frac{\pi}{4m}\right)\right\}\Big/\pi ab,$$

where $l =$ perimeter.

If m is infinite, $l = 2\pi r$; then $p = \left[2r(a+b-2r)\right]/ab$.

If b is infinite, $p = l/l'$ where $l' = \pi a =$ circumference of circle touching two parallel lines (see Quest. 1274).

731. (AMICUS.)—Find three numbers, such that the sum of every two of them may be a square number, and that the sum of the three squares thus found may be also a square number.

Solution by ARTEMAS MARTIN, LL.D.; Professor ZERR; *and others.*

Denote the required numbers by x, y, z; then must

$x+y = \square = p^2$ ... (1), $y+z = \square = q^2$ ... (2), $x+z = \square = r^2$ ... (3),

and $p^2+q^2+r^2 = \square = w^2$ ... (4).

Assume $p = 2st(u^2+v^2)$, $q = 2uv(s^2-t^2)$, $r = (s^2-t^2)(u^2-v^2)$;

then $w = (s^2+t^2)(u^2+v^2)$,

and (4) is satisfied. See *Mathematical Magazine*, Vol. II., No. 5, p. 73.

Adding (1), (2), (3), $2x+2y+2z = p^2+q^2+r^2 = \square = w^2$ .... (5).

From (1), (2), (3), and (5) we find

$$x = \tfrac{1}{2}(p^2-q^2+r^2),\quad y = \tfrac{1}{2}(p^2+q^2-r^2),\quad z = \tfrac{1}{2}(q^2+r^2-p^2),$$

where the sum of any two of the squares p^2, q^2, r^2 must be greater than the third.

Take $s = 4$, $t = 1$, $u = 2$, $v = 1$, and we have, after dividing by 5^2,

$$8^2+9^2+12^2 = 17^2.$$

Now, to avoid fractions, take $p = 24$, $q = 18$, $r = 16$, and we find

$$x = 254, \quad y = 322, \quad z = 2.$$

12045. (V. J. Bouton, B.Sc.) — Express $\int_0^\infty e^{-x^4}\, dx$ by elliptic integrals, proving that the square of the given integral $= \frac{1}{4}\pi^{\frac{1}{2}} \mathrm{F}(\frac{1}{2}\pi, \frac{1}{2})$.

Solution by J. McMahon, M.A.; Professor Sarkar; *and others.*

Putting $x^4 = z$, it becomes $\frac{1}{4}\int_0^\infty e^{-z} z^{-\frac{3}{4}}\, dz$, which is $\frac{1}{4}\Gamma(\frac{1}{4})$;

but $$[\Gamma(\tfrac{1}{4})]^2 = 2(2\pi)^{\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\cos^{-\frac{1}{2}}\theta\, d\theta = 4\pi^{\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{(1-\frac{1}{2}\sin^2\phi)^{\frac{1}{2}}}$$

[Williamson, Arts. 121–2, $2^{\frac{1}{2}}\sin\frac{1}{2}\theta = \sin\phi$.]

12041. (W. J. Greenstreet, M.A.)—Sum to n terms the series whose rth terms are respectively

$$\frac{2r-1}{(4r-3)(4r+1)(4r+5)}, \quad (2r-1)(2r+1)(2r+3)\sin r\theta,$$
$$\{(3r-2)(3r+1)(3r+4)\}^{-1}, \quad \{\cos r\theta\cos(r+1)\theta\}^{-1}.$$

Solution by H. J. Woodall, A.R.C.S.; Professor Zerr; *and others.*

1. $(2r-1)/\{(4r-3)(4r+1)(4r+5)\}$
$$= \tfrac{1}{2}\{(4r+5)-7\}/\{(4r-3)(4r+1)(4r+5)\}$$
$$= \tfrac{1}{2}\left[\tfrac{1}{4}\{(4r-3)^{-1}-(4r+1)^{-1}\} - \tfrac{7}{8}\{(4r-3)^{-1}(4r+1)^{-1}-(4r+1)^{-1}(4r+5)^{-1}\}\right].$$

Summing, we get
$$S = \tfrac{1}{2}\left[\tfrac{1}{4}\{1-(4n+1)^{-1}\}-\tfrac{7}{8}\{\tfrac{1}{5}-(4n+1)^{-1}(4n+5)^{-1}\}\right]$$
$$= \tfrac{3}{80}-\tfrac{1}{16}(8n+3)\{(4n+1)(4n+5)\}^{-1}.$$

2. $\{(2r-1)(2r+1)(2r+3)\sin r\theta\}\times 2\sin\frac{1}{2}\theta =$ (after reduction)
$$1.3.5\cos\tfrac{1}{2}\theta-(2n-1)(2n+1)(2n+3)\cos(n+\tfrac{1}{2})\theta+6\{3.5\cos\tfrac{3}{2}\theta + \&c.$$
$$+(2n-1)(2n+1)\cos(n-\tfrac{1}{2})\theta\},$$

and so for this series, reducing at each operation the degree of the coefficients of the circular functions.

Finally, we get $\qquad S\,(2\sin\tfrac{1}{2}\theta)^4$

$$= \left\{1\,.\,3\,.\,5\cos\tfrac{1}{2}\theta-(2n-1)(2n+1)(2n+3)\cos(n+\tfrac{1}{2})\,\theta\right\}(2\sin\tfrac{1}{2}\theta)^3$$
$$-6\left\{3\,.\,5\sin\theta-(2n-1)(2n+1)\sin n\theta\right\}(2\sin\tfrac{1}{2}\theta)^2$$
$$-6\times4\left\{5\cos1\tfrac{1}{2}\theta-(2n-1)\cos(n-\tfrac{1}{2})\,\theta\right\}(2\sin\tfrac{1}{2}\theta)$$
$$+6\times4\times2\left\{\sin2\theta-\sin(n-1)\,\theta\right\}.$$

3. $\left\{(3r-2)(3r+1)(3r+4)\right\}^{-1}$
$$= \tfrac{1}{6}\left[\left\{(3r-2)^2(3r+1)\right\}^{-1}-\left\{(3r+1)(3r+4\right\}^{-1}\right].$$
$$\text{Sum} = \tfrac{1}{6}\left[1/(1\,.\,4)-1/\left\{(3n+1)(3n+4)\right\}\right].$$

4. $\left\{\cos r\theta\,.\cos(r+1)\,\theta\right\}^{-1} = \sec r\theta\sec(r+1)\,\theta\,.\left[\sin\left\{(r+1)-r\right\}\theta/\sin\theta\right]$
$$= \operatorname{cosec}\theta\left\{\tan(r+1)\,\theta-\tan r\theta\right\}.$$
$$\text{Sum} = \operatorname{cosec}\theta\left\{\tan(n+1)\,\theta-\tan\theta\right\}.$$

12297. (Professor BHATTACHARYA, M.A.) — An indefinitely large plane area is ruled with parallel equidistant straight lines; a is the distance between two consecutive lines. A second set of parallel equidistant straight lines crosses the former set at right angles; b is the distance between two consecutive lines of the second set. A regular polygon is then thrown down on the area; the polygon has $4m$ sides, and the diameter of the circle circumscribing the polygon is less than a and also less than b. Find the chance that the polygon will fall across a line.

Solution by D. BIDDLE.

Let k = the perimeter of the polygon. Then, as shown in Quest. 10219 (Vol. LII.), the chances of its falling across the respective lines are $k/(\pi a)$ and $k/(\pi b)$. Consequently, the required probability

$$= 1-\left\{1-k/(\pi a)\right\}\left\{1-k/(\pi b)\right\}.$$

Now $k = 8mr\,.\sin\left\{\pi/(4m)\right\}$, where r = radius of circumcircle.

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769. (J. W. ELLIOTT.)—A quantity of corn is to be divided amongst n persons, and is calculated to last a certain time, if each person receives a peck every week; but, during the distribution, it is found that one person dies every week, and the corn in consequence lasts *twice* as long as was expected; required the quantity of corn, and the time it lasted.

Solution by Artemas Martin, LL.D.

Let t = time the corn was expected to last; then nt = pecks of corn;

$$n+(n-1)+(n-2)+(n-3)+\dots+\{n-(2t-1)\} = t(2n-2t+1)$$
$$= \text{number of pecks of corn consumed in the } 2t \text{ weeks};$$
$$\therefore\; t(2n-2t+1) = tn, \quad \text{whence} \quad t = \tfrac{1}{2}(n+1);$$
$$\therefore\; nt = \tfrac{1}{2}n(n+1) = \text{the number of pecks of corn,}$$

and $$2t = n+1 = \text{number of weeks it lasted.}$$

12288. (S. Tebay, B.A.)—a, b, c are conterminous edges of a tetrahedron; α, β, γ the angles contained by bc, ca, ab; A, B, C the dihedral angles through a, b, c; A_1, A_2, A_3 the areas of the faces contained by bc, ca, ab; and V the volume of the tetrahedron; then

$$\frac{\sin A}{\sin\alpha} = \frac{\sin B}{\sin\beta} = \frac{\sin C}{\sin\gamma} = \tfrac{3}{4}\cdot\frac{abc}{A_1A_2A_3}V.$$

Solution by Rev. T. R. Terry, M.A.; Professor Chakrivarti; *and others.*

Let OP, OQ, OR be the three conterminous edges; then $A_3 = \tfrac{1}{2}ab\sin\gamma$, and perp. from R on plane POQ = OR sin ROQ sin B = $c\sin\alpha\sin B$;

therefore $$V = \tfrac{1}{6}abc\sin\alpha\sin\gamma\sin B;$$

therefore $\tfrac{3}{4}\dfrac{abcV}{A_1A_2A_3} = \dfrac{\sin B}{\sin\beta}$. Similarly the other two results.

11891. (Professor McMurchy.)—Without knowing the angles of a triangular prism, prove that, by observing the minimum deviation (2α, 2β, 2γ) of rays passing in the neighbourhood of the three angles, the refractive index (μ) is given by

$$\mu^3 - \Sigma\cos\alpha\,.\,\mu^2 + \Sigma\cos(\beta+\gamma)\,.\,\mu = \cos(\alpha+\beta+\gamma).$$

Solution by H. J. Woodall, A.R.C.S.; H. W. Curjel, B.A.; *and others.*

We have, by the usual theory of refraction,

$$\mu = \sin(\tfrac{1}{2}A+\alpha)/\sin\tfrac{1}{2}A, \quad \&c.,$$

whence $$\cot\tfrac{1}{2}A = (\mu-\cos\alpha)/\sin\alpha, \quad \&c.$$

But, by the trigonometry of triangles, we find

$$1 = \tan\tfrac{1}{2}B\tan\tfrac{1}{2}C + \tan\tfrac{1}{2}C\tan\tfrac{1}{2}A + \tan\tfrac{1}{2}A\tan\tfrac{1}{2}B.$$

Substituting and reducing, we obtain

$$\mu^3 - \mu^2\Sigma\cos\alpha + \mu\Sigma\cos(\beta+\gamma) = \cos(\alpha+\beta+\gamma).$$

11909. (EDITOR.)—Solve the equations $2(x-y)^2\{(x+y)^2+\ z\}=z^4$,
$$\{3x(x+y)-z^2\}/\{3y(x+y)-z^2\}=(4y-7)/(7-4x),$$
$4(x+y)z-z^2-16xy=0$, if x, y be rational, and z an integer.

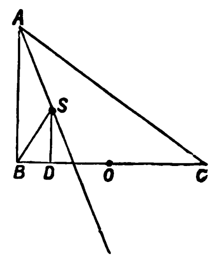
Solution by M. BRIERLEY; H. W. CURJEL, B.A.; *and others.*

The last equation gives $z=4x$ or $4y$. Substituting in the first equation $z=4x$, we get, as the only rational solution, $y=3x$. Substituting in the second equation, we get
$$\frac{12x^2-16x^2}{12x-7}=\frac{36x^2-16x^2}{7-4x},$$
whence $x=\frac{1}{2}$, $y=\frac{3}{2}$, $z=2$, or $x=0$, $y=0$, $z=0$.

Similarly, from $z=4y$, we get $y=\frac{1}{2}$, $x=\frac{3}{2}$, $z=2$.

11282. (R. KNOWLES, B.A.)—BC is a given line, O its mid-point; a variable point A it taken in BD at right angles to it; in the triangle ABC prove that (1) the envelope of the symmedian from A is a hyperbola having OB for its axis; (2) the locus of the symmedian point is a circle, which has OB for its diameter.

Solution by Professors ZERR, CHAKRIVARTI, *and others.*



Let S be the symmedian point. Then, with B as origin, the coordinates of A are (O, C) and of S $\left(\frac{ac^2}{2b^2}, \frac{a^2c}{2b^2}\right)$. Hence the equation to AS is
$$acy-ac^2+2b^2x=a^2x.$$

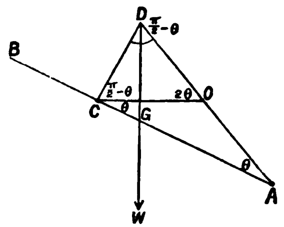
The envelope of this line subject to the condition $b^2-c^2=a^2$ is found by the usual method to be (remembering that b and c are variable and a constant) $8x^2-y^2-4ax=0$, a hyperbola on BO as axis.

2. Let $BS=r$. Then
$$r^2=\frac{a^2c^2}{4b^4}(a^2+c^2)=\frac{a^2c^2}{4b^2}=\frac{a}{2}\cdot r\cos\theta,$$
where $\angle SBO=\theta$; therefore $r=\frac{1}{2}a\cos\theta$, a circle on BO as diameter.

11881. (W. J. DOBBS.)—A uniform rod ACB, of weight W and length $4a$, rests upon a smooth peg C, and its lower end A is attached to

a fixed point O in the same horizontal line with C by means of a string OA. If OC = OA = c, show that the inclination of the rod to the horizon is $\cos^{-1}\left[\left\{a+(a^2+8c^2)^{\frac{1}{2}}\right\}/4c\right]$, and that no position of equilibrium is possible unless $a > c/6^{\frac{1}{2}}$ and $< c$.

Solution by Professors Zerr, Bhattacharya, *and others.*



(1) Let CA = x. In the position of equilibrium, the weight DG, the tension DO, and the reaction DC, perpendicular to AB, meet in a point D. Then

$$DO = OA = OC = c.$$

From triangle COA,

$$CA : AO = \sin 2\theta : \sin\theta;$$

$$\therefore\ x : c = 2\cos\theta : 1$$

or $$\cos\theta = x/(2c) \ldots\ldots\ldots\ldots (1).$$

Also, from triangle DOC,

$$DC = 2c\sin\theta.$$

In right triangle CGD, $CG = x-2a$, $CD = 2c\sin\theta$;

$$\therefore\ 2c\sin^2\theta = (x-2a)\cos\theta,$$

or $$\left\{2c(4c^2-x^2)\right\}/4c^2 = \left\{x(x-2a)\right\}/2c, \text{ from (1)};$$

$$\therefore\ x = \left\{a+(8c^2+a^2)^{\frac{1}{2}}\right\}/2; \quad \therefore\ \cos\theta = \left\{a+(8c^2+a^2)^{\frac{1}{2}}\right\}/4c;$$

$$\theta = \cos^{-1}\left[\left\{a+(8c^2+a^2)^{\frac{1}{2}}\right\}/4c\right].$$

2. The least value that x can have is $2a$, and the greatest $4a$;

$$\therefore\ 2a = \left\{a+(8c^2+a^2)^{\frac{1}{2}}\right\}/2, \text{ and } 4a = \left\{a+(8c^2+a^2)^{\frac{1}{2}}\right\}/2;$$

or $$a = c, \quad 6a^2 = c^2, \quad a = c/6^{\frac{1}{2}};$$

$\therefore\ a > c/6^{\frac{1}{2}}$ and $< c$, equilibrium is possible.

12104. (M. Brierley.)—Find two rational numbers, such that twice the square of their sum added to the square of their difference shall be a square.

Solution by T. Savage; Professor Zerr; *and others.*

Every square number is resolvable into sum of squares of two numbers with twice their product. Hence, if the result is a square, it must fulfil this condition. But result = (difference)^2 + (sum)^2 + (sum)^2; hence

$$(\text{sum})^2 = 2 \,.\, \text{difference} \,.\, \text{sum}, \quad \text{sum} = 2 \,.\, \text{difference}, \quad x+y = 2(x-y);$$

$$\therefore\ x = 3y.$$

Hence, any two rational numbers one of which is three times the other will satisfy the conditions.

APPENDIX.

UNSOLVED QUESTIONS.

3085. **(Professor Sylvester.)—1. Prove that the probability of five points taken arbitrarily within any given ellipsoid forming the angles of a reentrant polyhedron is**

$$\frac{3}{356\pi}\int_0^\pi\int_0^\pi d\theta\, d\phi\, (\sin\theta)\,(\cos\theta)^4\mu\nu,$$

where

$$\mu = \left\{1-(\sin\theta)^2(\sin\phi)^2\right\}^2+4(\sin\theta)^2(\cos\phi)^2-4(\sin\theta)^4(\sin\phi)^2(\cos\phi)^2,$$

and $$\nu = 3-3(\sin\theta)^2(\sin\phi)^2+8\sin\theta\sin\phi.$$

2. Show that this probability is the ratio of two commensurable numbers, and ascertain in value.

3089. (Professor Wolstenholme, M.A., D.Sc.)—**A large area is to be paved with circular discs of black marble which are to touch; the interstices to be filled up with slabs of white marble. Each piece of pavement must** be cut from a square slab; if the price of the slabs per foot of area vary as the length of a side, and if the cost of cutting the marble be a shillings per linear foot, and the price of the slab one foot square be b shillings; show that in order that the area may be paved with least expense the radius of each disc must be $\left(\frac{2\pi b}{a}\right)^{\frac{1}{2}}$.

3094. (Rev. F. D. Thomson, M.A.)—A heavy uniform rod, length $2a$, slips down with its extremities in contact with a smooth horizontal floor and a smooth vertical wall, *not* being initially in a plane perpendicular to the wall. Show that, if θ be the inclination to the vertical, ψ the inclination of the horizontal projection of the rod to the common section of the planes, the motion is determined by the equations

$$4\frac{d^2}{dt^2}(\cos\theta) = \cot\theta\ \sec\psi\frac{d^2}{dt^2}(\sin\theta\ \cos\psi) - 3\frac{g}{a},$$

$$4\frac{d^2}{dt^2}(\sin\theta\sin\psi) = \tan\psi\ \frac{d^2}{dt^2}(\sin\theta\ \cos\psi),$$

and deduce a first integral.

3102. (Professor Hudson, M.A.)—If the path of a ray cut at a constant angle α the surfaces of equal density in a variable medium, prove that $\mu = \mu_0\epsilon^{\phi\tan\alpha}$, where ϕ is the inclination of the path to a fixed line.

3106. (Professor Sylvester.)—1. A body is bounded above and below respectively by segments of two confocal ellipsoids having a common fixed centre. Prove that it may be made to rotate about that centre so as *to remain in contact* with *any two fixed parallel* planes bisecting the two segments respectively.

2. If the lower portion of the body containing a fixed point be given,

find the form of the upper portion in order that motion, as in (1), may be possible about that point and between any two fixed parallel tangent planes.

3117. (Rev. A. F. Torry, M.A.)—A rod is placed with one extremity at the middle point of the line joining two centres of force, which attract inversely as the square of the distance, the rod being at right angles to this line: find the velocity with which the centre of the rod will cross this line.

If the rod were placed at right angles to the line joining the centres, and very near the position of equilibrium, determine the time of its oscillation about that position.

3125. (Rev. T. P. Kirkman, M.A., F.R.S.)—N points are taken in the circumference of a circle; dark chords are drawn in any number or way each through non-consecutive points, and none meeting another within the circle; other faint chords also in any number or way are drawn each through two non-consecutive points, none meeting another faint chord within the circle. And this is done till no segment of the circle is possible which shall cut away one or more of the points without cutting one of the drawn chords. We have now a system of polygons having all their angles at the N points, and having for sides either dark chords with or withont arcs, or else faint chords with or without arcs. It is required to determine the number of different systems of polygons for $N = 7$, $N = 8$, $N = 9$, &c.; two systems being different only when no substitutions among the names $(a, b, c, \ldots)$ of the N angles can make them identical. The number of systems for $N = 7$ is less than 49.

3126. (Professor Cayley, F.R.S.)—Mr. Wolstenholme's Question 3067 may evidently be stated as follows:—

If (a, b, c) are the coordinates of a point on the cubic curve

$$a^3+b^3+c^3 = (b+c)(c+a)(a+b),$$

and if $$(b^2+c^2-a^2)x = (c^2+a^2-b^2)y = (a^2+b^2-c^2)z;$$

then (x, y, z) are the coordinates of a point on the same cubic curve.

This being so, it is required to find the geometrical relation of the two points to each other.

3130.—(Rev. A. F. Torry, M.A.)—A particle P describes an ellipse freely under the attraction of a second particle G which is constrained to move along the major axis; G (but not P) is attracted to the centre: find the laws of the attractions that PG may be always the normal at P. If it were conceivable that P should repel G with the same force that G attracts P, a certain relation between the masses of the two particles would render unnecessary any force to the centre.

3149. (A. Martin.)—A given right cone is cut by a random plane. What is the average area of the ellipse thus formed?

3160. (Rev. C. Taylor, D.D.)—If a normal meet the conic again in Q, and the directrices in R, R′; then (O being the pole of the chord, and S, S′ the foci) SR, OR′, and SR′, OR intersect on the normal at Q.

3162. (Professor Cayley, F.R.S.)—By a proper determination of the coordinates, the skew cubic through any six given points may be taken to be $x : y : z = y : z : w$; or, what is the same thing, the coordinates of the

six given points may be taken to be $(1, t_1, t_1^2, t_1^3) \dots (1, t_6, t_6^2, t_6^3)$ Assuming this, it is required to show that if

$$p_1 = \Sigma t_1,\ p_2 = \Sigma t_1 t_2,\ \dots\ \ p_6 = t_1 t_2 t_3 t_4 t_5 t_6,$$

and if $\nabla = 6xyzw - 4xz^3 - 4y^3w + 3y^2z^2 - x^2w^2$;

then the equation of the Jacobian surface of the six points is

$$\left.\begin{array}{l} 3\,(xp_3 \qquad\qquad + zp_1 - 2w)\ \delta_x \nabla \\ +\ (\qquad\qquad\quad 2zp_2 - wp_1)\ \delta_y \nabla \\ +\ (xp_5 - 2yp_4 \qquad\qquad\quad)\ \delta_z \nabla \\ +\ (2xp_6 - yp_5 \qquad\quad - wp_3)\ \delta_w \nabla \end{array}\right\} = 0.$$

3163. (Professor Sylvester.)—Let there be three simultaneous progressions of n terms each, say

$$(1)\ a_1 a_2 \dots a_n, \qquad (2)\ b_1 b_2 \dots b_n, \qquad (3)\ c_1 c_2 \dots c_n;$$

subject to the sole conditions that $a_r \,.\, b_r \,.\, c_r$ is positive and n constant. Use i, j, k to denote 1, 2, 3, in any order. Let p_i denote the number of permanancies of of sign in the ith progression; $p_i p_j$ the number of double permanencies $p_i v_j$ of permanence-variations, $v_i v_j$ of double variations, in the ith and jth progressions taken conjointly. Prove, and illustrate by an example, the following equations:—

$$\Delta\, p_i p_j = \tfrac{1}{2}(\Delta p_i + \Delta p_j + \Delta p_k),\ \ -\Delta\, v_i p_j = \tfrac{1}{2}(\Delta p_i - \Delta p_j + \Delta p_k),$$
$$\Delta\, v_i v_j = \tfrac{1}{2}(\Delta p_i - \Delta p_j + \Delta p_k).$$

[It will of course be understood that $v_i p_j$ refer to variations of sign in the ith progression associated with permanencies in the jth.]

3185. (Professor Cayley, F.R.S.)—An unclosed polygon of $(m+1)$ vertices is constructed as follows: viz., the abscissæ of the several vertices are $0, 1, 2, \dots m$, and corresponding to the abcsissa k, the ordinate is equal to the chance of $m+k$ heads in $2m$ tosses of a coin; and m then continually increases up to any very large value; what information in regard to the successive polygons, and to the areas of any portions thereof, is afforded by the general results of the Theory of Probabilities?

3208. (Professor Sir R. S. Ball, M.A.)—A rigid body capable of rotating around a fixed point is in staple equilibrium. If the body, when slightly displaced from its position by being rotated around an axis, continues for ever to vibrate around this axis, this line is called a normal axis. Prove that there are in general three normal axes; that when the forces have a potential, the three normal axes are conjugate diameters of the momental ellipsoid, and that they may be completely determined by a geometrical construction.

3211. (Professor Hudson, M.A.)—Prove that the curve

$$\frac{y}{b} = \left(\frac{x-a}{a}\right)^{\frac{5}{2}} + \frac{x^2}{c^2}$$

has a rampoid cusp at which the two branches have the same curvature; that the tangent at the cusp meets the curve again where $x = a\left(1 + \dfrac{a^4}{c^4}\right)$; and that it has a point of inflection where $x = a\left(1 + \dfrac{64}{225}\dfrac{a^4}{c^4}\right)$. Draw a figure representing the curve.

3219. (Rev. E. Hill, M.A.)—A cylindrical thin rigid surface contains fluid at a constant pressure P. Show that the tension at any point is given by $$\frac{d^2T}{d\phi^2}+T = P\frac{ds}{d\phi}.$$
Hence show that, if such a circular cylinder have a crack through it down a generating line, the tension opposite this crack is 2P × radius, and that the mutual action, normal to the surface, of the surfaces on each side of a generating line is a maximum 90° from the crack.

3229. (Professor Cayley, F.R.S.)—It is required to find the value of the elliptic integral $F(c, \theta)$ when c is very nearly $= 1$ and θ very nearly $= \frac{1}{2}\pi$; that is, the value of $\int_0^{\frac{1}{2}\pi-a} \frac{d\theta}{\{1-(1-b^2)\sin^2\theta\}^{\frac{1}{2}}}$, where a, b are each of them indefinitely small.

N.B.—Observe that, for $a = 0$, b small, the value is equal $\log 4/b$, and for $b = 0$, a small, the value is $-\log\cot\frac{1}{2}a$.

3231. (Professor Sir R. S. Ball, F.R.S.)—A rigid body acted upon by gravity rotates around a fixed point. Show that, when performing small oscillations, its movements are compounded of vibrations around two normal axes. Show that these normal axes are determined by the following construction. Draw the momental ellipsoid A; and draw another ellipsoid B, the axes of which are proportional to the squares of those of A, and in the same direction. Draw in ellipsoid A a plane conjugate to the vertical direction. This plane cuts A and B in two ellipses, the common conjugate diameters of which are the normal axes.

3234. (J. F. Moulton, M.A.)—An indefinitely thin shell in the form of an infinite elliptical cylinder, the section of the two surfaces being similar, similarly situated and concentric curves, is surrounded by homogeneous fluid which it attracts. Show that the pressure at any point is given by $\frac{m\rho}{2}\log\frac{a+b}{a'+b'}$, where a', b', a, b are the semi-axes of the confocal cylinders through the point and bounding the fluid respectively.

3235. (Professor Hudson, M.A.)—Find the shape of a uniform wire such that the moment of inertia of any portion of it bounded by two radii vectors about an axis through the pole perpendicular to its plane may vary as the angle between them.

3236. (A. Martin, LL.D.)—A circle is drawn with its centre in the surface of a given circle, and its circumference intersecting the circumference of the given circle; find the average area common to both circles.

3238. (Rev. E. Hill, M.A.)—An opening in the vertical dam of a large reservoir, always full, is covered by a sluice gate, sliding up and down in vertical grooves. Show that the necessary form of the opening in order that, when the gate is raised to any height, the rate of efflux of the water may be proportional to that height, is $y^2x = a^3$.

3250. (Professor Sylvester.)—If S_i be used as the symbol of the sum of i-ary products, prove that, when $i<n$,

$$\Sigma S(a_2a_3\ldots a_n)\frac{(a_1-\alpha_1)(a_1-\alpha_2)\ldots(a_1-\alpha_n)}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)}$$
$$=S_{i+1}(a_1a_2\ldots a_n)-S_{i+1}(\alpha_1\alpha_2\ldots\alpha_n).$$

[For example, let $n=3$, $i=2$; then the theorem becomes

$$bc\frac{(a-\alpha)(a-\beta)(a-\gamma)}{(a-b)(a-c)}+ca\frac{(b-\alpha)(b-\beta)(b-\gamma)}{(b-a)(b-c)}+ab\frac{(c-\alpha)(c-\beta)(c-\gamma)}{(c-a)(c-b)}$$
$$=abc-\alpha\beta\gamma,\ \text{which is obviously true.]}$$

3253. (Professor G. B. Vose.) — There are two casks, one of which contains a gallons of wine, and the other b gallons of water. There are also two measures, one of which has a capacity of c gallons more than the other. The larger measure is filled from the first cask, and its contents poured into the other; the smaller measure is then filled from the second cask, and its contents poured into the first. The *smaller* measure is filled from the first cask, and its contents poured into the second, and the compound transfer is completed by filling the larger measure from the second cask and pouring into the first. After n compound transfers like the one described above, the quantity of wine remaining in the first cask is d gallons. Required the capacity of each of the two measures.

3254. (Professor Sir R. S. Ball, F.R.S.)—The centre of gravity of a rigid body, which is performing small oscillations about a fixed point under the action of gravity, has a movement compounded of two vibrations at right angles to each other.

3255. (Professor Clifford, F.R.S.)—Two planes A, B are said to have an (x, y) correspondence, when to every point on the plane A correspond y points on the plane B, and to every point on B correspond x points on A.

On each plane there is in general a locus of points, two of whose correspondents coincide; this is called the *cross-curve*. (Uebergangscurbe: Clebsch in *Math. Annalen.*) On each plane there is also a locus of these united correspondents; this is called the *node-curve*.

1. If a curve touch the cross-curve in either plane, the corresponding curve in the other plane will have a node lying on the node-curve in that plane.

2. The correspondence may be represented as a (1, 1) correspondence of two multiple planes A′, B′; A′ consisting of y sheets and B′ of x sheets, which are connected together along the cross-curves.

3. In a $(1, y)$ correspondence, in which to straight lines in the plane A correspond curves of deficiency p in the plane B, the order of the cross-curve in A is $=2(y+p-1)$.

3271. (Rev. E. Hill, M.A.)—A V-shaped trough, with edge horizontal, length finite, height indefinite, contains a known quantity of water. Into this a similar, equal, and similarly situated prism is being plunged with given uniform velocity. Assuming the hypothesis of parallel sections, find the pressure at any point of the fluid at any time.

3277. (A. Martin, LL.D.)—Three persons play at dice as follows: each has a die and is to throw it ten times, the one throwing the highest to be winner. A throws 3 times and turns up 15, B throws 4 times and turns

up 18, C throws 6 times and turns up 32. What now are the respective chances of the final success of each player?

3278. (Professor Cayley, F.R.S.)—It is required, with nine numbers each taken three times, to form nine triads containing twenty-seven distinct duads (or, what is the same thing, no duad twice), and to find in how many essentially distinct ways this can be done. (See Quest. 3206.)

3286. (A. Martin, LL.D.)—Three men throw each a die, the one that first turns up, in all, 40 spots to be winner. When A, B, C have thrown 15, 24, 28, respectively, it is required to determine each one's chance of final success, each having thrown the same number of times, and each having the same number of throws.

3291. (J. J. Walker, F.R.S.)—Referring to Quest. 3173, prove also that

$$\cot \mathrm{ADB} = \frac{\cot \mathrm{C} - \cot \mathrm{B}}{2 \cos \frac{1}{2} a} \ (4); \quad \frac{\tan \mathrm{AE}}{\tan \mathrm{DE}} = \frac{\sin b \sin c}{1 - \cos b \cos c} \ (5).$$

3293. (J. Griffiths, M.A.)—Find an expression for the radius (ρ) of curvature at any point (α, β, γ) of the curve $\mathrm{F}(\alpha, \beta, \gamma) = 0$.

3297. (A. B. Evans, M.A.)—Find the solidity of a spherical segment of given bases and altitude, the particles of which vary in density as the nth power of their distance from the centre of the sphere.

3298. (A. Martin, LL.D.)—Two equal ellipses have a common centre and their axes inclined at an angle ϕ; find the average of the area common to both, supposing all values of ϕ from 0 to $\frac{1}{2}\pi$ to be equally probable.

3305. (Professor Sylvester.)—Find the chance that if three points be taken at random inside a circle, any two of them shall be nearer to one another than the remaining one to the centre.

3318. (R. Tucker, M.A.)—Inscribe the maximum rectangle in a lemniscate.

3319. (Professor Hudson, M.A.)—A sphere is totally immersed; find the locus of the centres of the small circles which enclose portions of the surface on which the whole pressure is the same.

3320. (J. J. Walker, M.A.)—Show how to transform two binary cubics u, v by the same linear substitutions into $(a, b, c, d \between xy)^3$, and, to a factor, $(a, -b, c, -d \between xy)^3$ respectively, when this transformation is possible, viz., when the relation $\Delta\Theta'^2 - \Delta'\Theta^2 = 0$ is satisfied, the discriminant of $u + \lambda v$ being written $(\Delta, \Theta, \Phi, \Theta', \Delta' \between 1\lambda)^4$.

3321. (Rev. W. A. Whitworth, M.A.)—A regular "screw-polygon" is formed of equal straight lines. Any two adjacent sides contain an angle $\pi - \theta$, and the planes containing any side and either of the adjacent sides contain an angle ψ. Show that, if there be n sides in a convolution,

$$\cos \pi/n = \cos \tfrac{1}{2}\theta \cos \tfrac{1}{2}\psi.$$

3331. (C. W. Merrifield, F.R.S.)—A billiard table is lighted by two burners. How must they be placed so as to secure the table being as evenly illuminated as possible?

3336. (Professor Clifford, F.R.S.)—1. The sides of a triangle repel with a force varying inversely as the cube of the distance; find the position in which a particle will rest.

2. Also, supposing the faces of a tetrahedron to repel according to the same law; find where a particle will rest.

3337. (W. S. Burnside, M.A.)—Find the locus of points such that the polar conics with reference to the curve U shall be equilateral hyperbolas, where $U = \Sigma \dfrac{\sin 2A}{xyz + x^2(-x\cos A + y\cos B + z\cos C)}$;

and show that this locus passes through the vertices of the triangle ABC, and through the feet of the perpendiculars of the same triangle.

3340. (A. B. Evans, M.A.)—Two points are taken at random in a given equilateral triangle, and a straight line is drawn at random across the triangle. Find the probability that the line separates the points.

3341. (S. Watson.) — From any point P in the curve of a given ellipse, two lines are drawn through the foci F, f to meet the curve again in Q and R. Find the average area of the triangle PQR.

3359. (Rev. F. D. Thomson, M.A.)—At the election of a School Board of nine members there are twelve candidates. Show that each elector may dispose of his nine votes in 92,378 different ways.

3365. (S. Watson.)—From any point P in the circumference of a given circle a chord PQ is drawn at random; and upon it two points R, S are taken at random. What is the chance that the circle upon RS as diameter will be wholly upon the given circle?

3381. (Professor Cayley, F.R.S.)—Show that the attraction of an indefinitely thin double convex lens on a point at the centre of one of its faces is equal to that of the infinite plate included between the tangent plane at the point and the parallel tangent plane of the other face of the lens.

3386. (J. J. Walker, F.R.S.)—If normals to the ellipse $b^2x^2 + a^2y^2 - a^2b^2 = 0$ be drawn from any point on the curve $(a^2x^2 + b^2y^2 - c^4)^3 + 54a^2b^2c^4x^2y^2 = 0$, prove that they form an harmonic pencil.

3389. (S. Watson.)—Two circles (each radius r) are described on the semi-minor axes of an ellipse as diameters, and two others (each radius r) having their centres on the major axis to touch the former two, and the ellipse A is the area of the rhombus formed by joining the centres of the four circles, and A′ the area of that formed by tangents at the points of contact of the circles; then $r'/r = 2e$, and $R/R' = 1 + e$.

3390. (A. B. Evans, M.A.)—Find the area of the maximum ellipse that can be inscribed in the quadrant of a given circle.

3392. (Rev. H. T. Sharpe, M.A.)—One end of a heavy rod rests on a horizontal plane and against the foot of a vertical wall, the other end rests against a parallel vertical wall, all the surfaces being smooth. Show that, if it slips down, the angle ϕ through which it turns round the common normal to the vertical walls is given by the equation

$$\frac{d\phi^2}{dt^2}(1 + 3\cos^2\phi) = -\frac{6g}{\sqrt{(a^2 - b^2)}}\sin\phi + C,$$

where $2a$ is the length of the rod, and $2b$ the distance between the walls.

3398. (Rev. A. F. Torry, M.A.)—Parallel rays fall on one side of a refracting sphere ($\mu = \frac{2}{3}\sqrt{3}$), and form a caustic surface within the sphere. Find the length of that portion of the arc of its generating curve which lies within the sphere.

3399. (Rev. W. A. Whitworth, M.A.)—Two particles are approaching a pole by the same curvilinear path, and the rate at which each describes vectorial areas about the pole is measured by half the (constant) rectangle contained by the radius vector and polar subtangent. Show that the angular distance between the particles, viewed from the pole, is multiplied in every unit of time by e^2.

3403. (T. Cotterill, M.A.)—1. If $(x_1 y_1)$, $(x_2 y_2)$ are the perpendiculars from two conjugate foci of a conic upon any two of its conjugate lines x and y, then $(x_1 y_2 + y_1 x_2) \sec(xy)$ is invariable.

2. Hence (or geometrically), conjugate foci of a conic touching CA, CB at A and B are foci of a conic touching AB and the reflexions of AB to CA and CB.

3. The same holds good for the sphere.

3423. (Editor.)—AB is the vertical diameter of a circle. A ball descends down the chord AC, and, being reflected by the plane BC, describes its path as a projectile. Find the average range of the ball on the diameter CD; supposing all coefficients of friction relative to the descent of the ball on the chord to exist, for which motion is possible.

3424. (Professor Genese, M.A.)—The product of the two normals that can be drawn to a parabola from a point on it = 2SP . PG.

3425. (T. Mitcheson.)—A body which floats in two fluids, which touch each other but do not mix, is found to have volumes v_1, v_2, v_3 in the uppermost fluid, when the densities of the latter are s_1, s_2, s_3 respectively. If the body is just below the surface of the uppermost fluid, prove that $v_1 s_1 = v_2 s_2 = v_3 s_3$.

3426. (Professor Cayley, F.R.S.)—Show that the envelope of a variable circle having its centre on a given conic and cutting at right angles a given circle is a bicircular quartic; which, when the given conic and circle have double contact, becomes a pair of circles: and, by means of the last-mentioned particular case of the theorem, connect together the porisms arising out of the two problems—

(1) Given two conics, to find a polygon of the n sides inscribed in the one and circumscribed about the other.

(2) Given two circles, to find a closed series of n circles each touching the two circles and the two adjacent circles of the series.

3443. (Professor Hudson, M.A.)—Trace the curves

$$2\left\{x^2-(y-a)^2\right\}\left\{4x^2-(y-2a)^2\right\}+9y^2\left\{x^2+a(y-a)\right\}=0,$$
$$\alpha(\alpha^2-\beta\gamma)(\beta+\gamma)=\beta^2\gamma^2.$$

3446. (Editor.)—Two circles are drawn at random on the surface of the earth; find the probability that they will lie on the same meridian.

3450. (Rev. E. Hill, M.A.)—The radius of a thermometer bulb is ·3 of an inch, that of the stem ·01, height of the freezing point above the bulb, 1 inch, and of the boiling point $7\frac{1}{2}$ inches. Show that the expansion of mercury for a degree centigrade is about $\frac{1}{5550}$ of its volume.